Ch. 13

13.1

Parametric eqns trace a specific direction as $t$ increases

This direction is called the orientation or direction of increasing parameter

The curve together w/ its orientation is called the graph

Given $\vec{r} = \vec{r}(t)$, this fn produces a vector

$\vec{r} = \vec{r}(t)$ defines $\vec{r}$ as a vector-valued fn of a real variable, or simply a vector-valued fn.

If $\vec{r}(t)$ is a vector-valued fn in 2-sp, it can be represented as

$$\vec{r} = \vec{r}(t) = \langle x(t), y(t) \rangle = x(t)\vec{i} + y(t)\vec{j}$$

$\vec{r}(t)$ defines a pair of real-valued fns; these are the component fns

The domain of a vector-valued fn $\vec{r}(t)$ is the set of allowable values of $t$.

P. 864

7) $x = 2t,\ y = 2\sin 3t,\ z = 5\cos 3t$

$\vec{r}(t) = 2t\vec{i} + 2\sin 3t\,\vec{j} + 5\cos 3t\,\vec{k}$

18) $\vec{r} = -2\vec{i} + t\vec{j} + (t^2 - 1)\vec{k}$

$x = -2 \quad y = t \quad z = t^2 - 1$

$z = y^2 - 1$

parabola in $x = -2$ plane

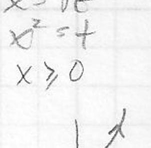
$v(-2, 0, -1)$ opens up

30) $\vec{r}(t) = \sqrt{t}\,\vec{i} + (2t + 4)\vec{j}$

$x = \sqrt{t} \quad y = 2t + 4$

$x^2 = t$

$x \geq 0 \quad y = 2x^2 + 4$



13.2

$\lim_{t \to a} \vec{r}(t) = \vec{L}$

if given $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni$ $\|\vec{r}(t) - \vec{L}\| < \epsilon$ if $0 < |t-a| < \delta$

A vector-valued fn $\vec{r}(t)$ is continuous @ $t = c$ if
(a) $\vec{r}(c)$ is defined

(b) $\lim_{t \to c} \vec{r}(t)$ exists

(c) (a) = (b)

Derivatives & Integrals

$$\vec{r}\,'(t) = \lim_{w \to t} \frac{\vec{r}(w) - \vec{r}(t)}{w - r}$$

Dot product: $\frac{d}{dt}\left(\vec{r}_1(t) \cdot \vec{r}_2(t) = \vec{r}_1(t) \cdot \frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt} \cdot \vec{r}_2(t)\right.$

Cross product: $\frac{d}{dt}\left[\vec{r}_1(t) \times \vec{r}_2(t)\right] = \vec{r}_1(t) \times \frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt} \times \vec{r}_2(t)$

Note: $\vec{r}(t) \cdot \vec{r}\,'(t) = 0 \Rightarrow \vec{r}(t)$ & $\vec{r}\,'(t)$ are orthogonal

$$\frac{d}{dt}\left[\int \vec{r}(t)\,dt\right] = \vec{r}(t)$$

$$\int \vec{r}\,'(t)\,dt = \vec{r}(t) + c$$

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

Vector eqn of the tangent line $\vec{r} = \vec{r}_0 + t\vec{v}_0$

13.2

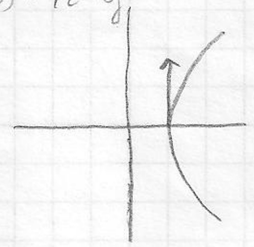
P874

17) $\vec{r}(t) = \sec t\,\vec{i} + \tan t\,\vec{j}$ ; $t_0 = 0$

$\vec{r}\,'(t) = \sec t \tan t\,\vec{i} + \sec^2 t\,\vec{j}$

$\vec{r}\,'(0) = 0\vec{i} + 1\vec{j}$

$\vec{r}(0) = 1\vec{i} + 0\vec{j}$



24) $\vec{r}(t) = e^{2t}\vec{i} - 2\cos 3t\,\vec{j}$ ; $t_0 = 0$

$\vec{r}(0) = 1\vec{i} - 2\vec{j}$ $\quad \vec{r}\,'(t) = 2e^{2t}\vec{i} + 6\sin t\,\vec{j}$

$\vec{v}(0) = \vec{r}(0) = 2\vec{i} + 0\vec{j}$

$\therefore$ tangent line $\vec{r} = 1\vec{i} - 2\vec{j} + t(2\vec{i} + 0\vec{j})$

$\vec{r} = (1+2t)\vec{i} + (-2+0t)\vec{j}$

and

$x = 1+2t,\ y = -2$

31b) $\lim_{t\to 0} \left(\vec{r}(t) \times \vec{r}\,'(t)\right)$

$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + \vec{k}$

$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + 0\vec{k}$

$$\vec{r} \times \vec{r}\,' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t & \sin t & 1 \\ -\sin t & \cos t & 0 \end{vmatrix}$$

$= -\cos t\,\vec{i} + \sin t\,\vec{j} + \vec{k}$

$\lim_{t\to 0} (\vec{r} \times \vec{r}\,') = -1\vec{i} + 0\vec{j} + \vec{k}$

38) $\int \left(te^{t}\vec{i} + \ln t\,\vec{j}\right) dt$

$(te^t - e^t)\vec{i} + (t\ln t - t)\vec{j} + C$

| | $u$ | $dv$ |
|---|---|---|
| + | $t$ | $e^t$ |
| − | $1$ | $e^t$ |
| + | $0$ | $e^t$ |

| | $u$ | $dv$ |
|---|---|---|
| + | $\ln t$ | $dt$ |
| − | $\frac{1}{t}$ | $t$ |

53a) $\vec{r} = t\vec{i} + t^2\vec{j} - 3t\vec{k}$

intersect $2x - y + z = -2$

$2(t) - 1(t^2) + 1(-3t) = -2$

$2t - t^2 - 3t = -2$

$t^2 + t - 2 = 0$

$(t+2)(t-1) = 0 \quad t = -2, 1$

$t = -2 \qquad t = 1$

$P_1(-2, 4, 6) \quad P_2(1, 1, -3)$

b) $\vec{r}\,' = \vec{i} + 2t\vec{j} - 3\vec{k}$ , $\vec{n} = \langle 2, -1, 1\rangle$

$t = -2 \quad \vec{r}\,' = \langle 1, -4, -3\rangle$

$\cos\theta = \dfrac{|\vec{r}\,' \cdot \vec{n}|}{\|\vec{r}\,'\|\|\vec{n}\|} = \dfrac{3}{\sqrt{26}\sqrt{6}} = \dfrac{3}{\sqrt{156}}$

$\theta \approx 76^\circ$

$t = 1 \quad \vec{r}\,' = \langle 1, 2, -3\rangle$

$\cos\theta = \dfrac{|-3|}{\sqrt{14}\sqrt{6}} = \dfrac{3}{\sqrt{84}}$

$\theta \approx 71^\circ$

13.3

$\vec{r}(t)$ is smoothly parametrized if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq \vec{0}$

a smoothly parametrized fn is said to have a continuously turning tangent vector.

Arc Length

2-space $x = x(t),\ y = y(t) \quad a \le t \le b$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$ (from section 7.4)

3-space $x = x(t),\ y = y(t),\ z = z(t);\ a \le t \le b$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

Arc Length, vector-valued fns

C is graph in 2- or 3-sp of a smooth vector-valued fn $\vec{r}(t)$, then

$$L = \int_a^b \left\| \frac{d\vec{r}}{dt} \right\| dt$$ t goes from a to b

A parametric representation of a curve w/ arc length as the parameter is called an arc length parametrization

A change of parameter in a v-v fn $\vec{r}(t)$ is a substitution $t = g(\tau)$ that produces a new v-v fn $\vec{r}[g(\tau)]$ having the same graph as $\vec{r}(t)$, but possibly traced differently as $\tau$ increases.

$$\frac{d\vec{r}}{d\tau} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{d\tau}$$

$$s = \int_{t_0}^{t} \left\| \frac{d\vec{r}}{du} \right\| du$$

this can be expressed in component form as

2-sp $$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\ du$$

3-sp $$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\ du$$

$$\left\| \frac{d\vec{r}}{dt} \right\| = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\left\| \frac{d\vec{r}}{dt} \right\| = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$\left\| \frac{d\vec{r}}{ds} \right\| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} = 1$$

$$\left\| \frac{d\vec{r}}{ds} \right\| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = 1$$

**P. 883**

4) $\vec{r}(t) = \cos t^2\,\vec{i} + \sin t^2\,\vec{j} + e^{-t}\,\vec{k}$

$\vec{r}\,'(t) = -2t\sin t^2\,\vec{i} + 2t\cos t^2\,\vec{j} - e^{-t}\,\vec{k}$

smooth

10) $x = \frac{1}{2}t,\ y = \frac{1}{3}(1-t)^{3/2},\ z = \frac{1}{3}(1+t)^{3/2};$
$-1 \le t \le 1$

$$L = \int_{-1}^{1} \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(-\tfrac{1}{2}(1-t)^{\frac{1}{2}}\right)^2 + \left(\tfrac{1}{2}(1+t)^{\frac{1}{2}}\right)^2}\, dt$$

$$L = \int_{-1}^{1} \sqrt{\tfrac{1}{4} + \tfrac{1}{4}(1-t) + \tfrac{1}{4}(1+t)}\, dt$$

$$L = \int_{-1}^{1} \sqrt{\tfrac{1}{4}[1 + 1 - \cancel{t} + 1 + \cancel{t}]}\, dt$$

$$L = \int_{-1}^{1} \frac{\sqrt{3}}{2}\, dt = \frac{\sqrt{3}}{2} t \Big|_{-1}^{1} = \sqrt{3}$$

13) $\vec{r}(t) = 3\cos t\,\vec{i} + 3\sin t\,\vec{j} + t\,\vec{k};\ 0 \le t \le 2\pi$

$$L = \int_0^{2\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (1)^2}\, dt$$

$$L = \int_0^{2\pi} \sqrt{9(\cos^2 t + \sin^2 t) + 1}\, dt$$

$$L = \int_0^{2\pi} \sqrt{10}\, dt = \sqrt{10}\, t \Big|_0^{2\pi} = 2\pi\sqrt{10}$$

16) $\vec{r} = 3\cos t\,\vec{i} + 3\sin t\,\vec{j};\ t = \pi\tau$

$$\frac{d\vec{r}}{d\tau} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{d\tau}$$

$$= (-3\sin t\,\vec{i} + 3\cos t\,\vec{j})(\pi)$$

$$= -3\pi\sin \pi\tau\,\vec{i} + 3\pi\cos\pi\tau\,\vec{j}$$

22) $x = -5 + 3t,\ y = 2t,\ z = 5 + t$

same direction as given line, ref. $(-5, 0, 5)$

a) $\vec{r}(t) = \langle -5, 0, 1 \rangle$ when $t = 0$ so

$$s = \int_0^t \sqrt{(3)^2 + (2)^2 + (1)^2}\, dt$$

$$s = \int_0^t \sqrt{9 + 4 + 1}\, dt$$

$$s = \int_0^t \sqrt{14}\, dt = \sqrt{14}\, t$$

$$s = \sqrt{14}\, t \Rightarrow t = \frac{s}{\sqrt{14}}$$

$$\therefore x = -5 + \frac{3s}{\sqrt{14}},\ y = \frac{2s}{\sqrt{14}},\ z = 5 + \frac{s}{\sqrt{14}}$$

b) $$r(s)\Big|_{s=10} = \left\langle -5 + \frac{30}{\sqrt{14}}, \frac{20}{\sqrt{14}}, 5 + \frac{10}{\sqrt{14}} \right\rangle$$

28) $\vec{r}(t) = \sin e^t\,\vec{i} + \cos e^t\,\vec{j} + \sqrt{3}\,e^t\,k;\ t \ge 0$

$$s = \int_0^t \sqrt{(e^t \cos e^t)^2 + (-e^t \sin e^t)^2 + (\sqrt{3}e^t)^2}\, dt$$

$$s = \int_0^t \sqrt{e^{2t}(\cos^2 e^t + \sin^2 e^t) + 3e^{2t}}\, dt$$

$$s = \int_0^t \sqrt{4e^{2t}}\, dt = \int_0^t 2e^t\, dt = 2e^t - 2$$

$$s = 2(e^t - 1)$$

$$\frac{s}{2} + 1 = e^t$$

$$\therefore x = \sin\left(\frac{s}{2} + 1\right),\ y = \cos\left(\frac{s}{2} + 1\right),\ z = \sqrt{3}\left(\frac{s}{2} + 1\right)$$

32) $r = r(t)$, $\theta = \theta(t)$, $z = z(t)$ for $a \le t \le b$

$x = r\cos\theta$ , $y = r\sin\theta$

$\frac{dx}{dt} = \cos\theta \frac{dr}{dt} - r\sin\theta \frac{d\theta}{dt}$

$\frac{dy}{dt} = \sin\theta \frac{dr}{dt} + r\cos\theta \frac{d\theta}{dt}$

$\frac{dz}{dt} = \frac{dz}{dt}$

$L = \int_a^b \sqrt{\left[\cos\theta \frac{dr}{dt} - r\sin\theta \frac{d\theta}{dt}\right]^2 + \left[\sin\theta \frac{dr}{dt} + r\cos\theta \frac{dr}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}\, dt$

$L = \int_a^b \sqrt{\cos^2\theta \left(\frac{dr}{dt}\right)^2 - 2r\cancel{\cos\theta\sin\theta \frac{dr}{dt}\frac{d\theta}{dt}} + r^2\sin^2\theta \left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{dr}{dt}\right)^2 + 2r\cancel{\cos\theta\sin\theta \frac{dr}{dt}\frac{d\theta}{dt}} + r^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$L = \int_a^b \sqrt{(\cos^2\theta + \sin^2\theta)\left(\frac{dr}{dt}\right)^2 + r^2(\sin^2\theta + \cos^2\theta)\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$L = \int_a^b \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

33a) $r = e^{2t}$, $\theta = t$, $z = e^{2t}$ ; $0 \le t \le \ln 2$

$L = \int_0^{\ln 2} \sqrt{(2e^{2t})^2 + r^2(1)^2 + (2e^{2t})^2}\, dt$

$L = \int_0^{\ln 2} \sqrt{4e^{4t} + e^{4t} + 4e^{4t}}\, dt$

$L = \int_0^{\ln 2} \sqrt{9e^{4t}}\, dt$

$L = \int_0^{\ln 2} 3e^{2t}\, dt$

$L = \frac{3}{2} e^{2t} \Big|_0^{\ln 2}$

$L = \frac{3}{2}\left[4 - 1\right] = \underline{\underline{\frac{9}{2}}}$

Ch. 13

13.4

Unit Tangent vector to C @ t $\quad \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

Unit Normal vector to C @ t $\quad \vec{N}(t) = \dfrac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$

If $\vec{r}$ is parametrized by the arc length, $\vec{r}(s)$

Unit Tangent vector becomes $\vec{T}(s) = \vec{r}\,'(s)$

Unit Normal vector becomes $\vec{N}(s) = \dfrac{\vec{r}\,''(s)}{\|\vec{r}\,''(s)\|}$

Binormal vector to C @ t is $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

$\vec{B}(t)$ is orthogonal to both $\vec{T}(t)$ & $\vec{N}(t)$ and is oriented relative to $\vec{T}(t)$ & $\vec{N}(t)$ by the right-hand rule.

Alternatively, $\vec{B}(t)$ can be expressed directly in term of $\vec{r}(t)$ as

$$\vec{B}(t) = \frac{\vec{r}\,'(t) \times \vec{r}\,''(t)}{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}$$

In the case where the parameter is arc length,

$$\vec{B}(s) = \frac{\vec{r}\,'(s) \times \vec{r}\,''(s)}{\|\vec{r}\,''(s)\|}$$

At each point on a smooth parametric curve C in 3-sp, these vectors determine three mutually perpendicular planes that pass through the point:

① The Rectifying Plane ($\vec{T}\vec{B}$-plane)
② The Osculating Plane ($\vec{T}\vec{N}$-plane)
③ The Normal Plane ($\vec{N}\vec{B}$-plane)

P. 889

4) $\vec{r}(t) = \frac{1}{2}t^2\vec{i} + \frac{1}{3}t^3\vec{j}\ ;\ t=1$

$\vec{r}\,'(t) = t\,\vec{i} + t^2\vec{j}$

$\|\vec{r}\,'(t)\| = \sqrt{t^2 + t^4}$

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

$\vec{T}(t) = (t^2+t^4)^{-1/2}(t\vec{i} + t^2\vec{j})$

$\vec{T}(1) = \underline{\dfrac{1}{\sqrt{2}}\vec{i} + \dfrac{1}{\sqrt{2}}\vec{j}}$

$\vec{T}\,'(t) = (t^2+t^4)^{-1/2}(\vec{i} + 2t\vec{j}) - (t+2t^3)(t^2+t^4)^{-3/2}(t\vec{i} + t^2\vec{j})$

$\vec{T}\,'(1) = \dfrac{1}{\sqrt{2}}\vec{i} + \dfrac{2}{\sqrt{2}}\vec{j} - \dfrac{3}{2\sqrt{2}}(\vec{i} + \vec{j}) = -\dfrac{1}{2\sqrt{2}}\vec{i} + \dfrac{1}{2\sqrt{2}}\vec{j}$

$\vec{N}(t) = \dfrac{T'(t)}{\|T'(t)\|} \Rightarrow \dfrac{T'(1)}{\|T'(1)\|} = \dfrac{\left(-\frac{1}{2\sqrt{2}}\vec{i} + \frac{1}{2\sqrt{2}}\vec{j}\right)}{\sqrt{\frac{1}{8} + \frac{1}{8}}} = \underline{-\dfrac{1}{\sqrt{2}}\vec{i} + \dfrac{1}{\sqrt{2}}\vec{j}}$

p.2 13.4

p. 889

17) $\vec{r}(t) = (\sin t - t\cos t)\vec{i} + (\cos t + t\sin t)\vec{j} + \vec{k}$

$$\vec{T}(t) = \frac{t\sin t\,\vec{i} + t\cos t\,\vec{j}}{t} = \underline{\sin t\,\vec{i} + \cos t\,\vec{j}}$$

$$\vec{N}(t) = \frac{\cos t\,\vec{i} - \sin t\,\vec{j}}{1}$$

$$\vec{B}(t) = \vec{T} \times \vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sin t & \cos t & 0 \\ \cos t & \sin t & 0 \end{vmatrix} = \underline{-\vec{k}}$$

20) $\vec{r}(t) = e^t\,\vec{i} + e^t\cos t\,\vec{j} + e^t\sin t\,\vec{k}\,;\ t=0$

$$\vec{T}(t) = \frac{e^t\,\vec{i} + (e^t\cos t - e^t\sin t)\vec{j} + (e^t\sin t + e^t\cos t)\vec{k}}{\sqrt{e^{2t} + e^{2t}(\cos^2 t - \cancel{2\cos t\sin t} + \sin^2 t) + e^{2t}(\sin^2 t + \cancel{2\cos t\sin t} + \cos^2 t)}}$$

$$\vec{T}(t) = \frac{e^t\,\vec{i} + (e^t\cos t - e^t\sin t)\vec{j} + (e^t\sin t + e^t\cos t)\vec{k}}{\sqrt{e^{2t}[1+1+1]}}$$

$$\vec{T}(t) = \frac{\cancel{e^t}\left[\vec{i} + (\cos t - \sin t)\vec{j} + (\sin t + \cos t)\vec{k}\right]}{\sqrt{3}\,\cancel{e^t}}\Bigg|_{t=0} \quad \vec{T}(0) = \underline{\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}}$$

$$\vec{N}(t) = \frac{\left[0\vec{i} - (\sin t + \cos t)\vec{j} + (\cos t - \sin t)\vec{k}\right]}{\sqrt{\sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t + \cos^2 t - \cancel{2\sin t\cos t} + \sin^2 t}}$$

$$\vec{N}(t) = \frac{\left[0\vec{i} - (\sin t + \cos t)\vec{j} + (\cos t - \sin t)\vec{k}\right]}{\sqrt{2}}\Bigg|_{t=0} = 0\vec{i} \underline{- \frac{1}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}}$$

$$\vec{B}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{3}} & \frac{\cos t - \sin t}{\sqrt{3}} & \frac{\sin t + \cos t}{\sqrt{3}} \\ 0 & \frac{-(\sin t + \cos t)}{\sqrt{2}} & \frac{(\cos t - \sin t)}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{6}}\left[(\cos t - \sin t)^2 + (\sin t + \cos t)^2\right]\vec{i} - \frac{1}{\sqrt{6}}\left[\cos t - \sin t\right]\vec{j} - \frac{1}{\sqrt{6}}(\sin t + \cos t)\vec{k}\Bigg|_{t=0}$$

$$= \underline{\frac{2}{\sqrt{6}}\vec{i} - \frac{1}{\sqrt{6}}\vec{j} - \frac{1}{\sqrt{6}}\vec{k}}$$

$\vec{r}(0) = \vec{i} + \vec{j} = \langle 1, 1, 0 \rangle$

Osculating ($\vec{T}\vec{N}$-plane)

$\frac{2}{\sqrt{6}}(x-1) - \frac{1}{\sqrt{6}}(y-1) - \frac{1}{\sqrt{6}}(z-0) = 0$

$\underline{2x - y - z = 1}$

Normal ($\vec{N}\vec{B}$-plane)

$\frac{1}{\sqrt{3}}(x-1) + \frac{1}{\sqrt{3}}(y-1) + \frac{1}{\sqrt{3}}(z-0) = 0$

$\underline{x + y + z = 2}$

Rectifying ($\vec{T}\vec{B}$-plane)

$0(x-1) - \frac{1}{\sqrt{2}}(y-1) + \frac{1}{\sqrt{2}}(z-0) = 0$

$\underline{-y + z = -1}$

For a curve in 2- or 3-space the "sharpness" of the bend in C is closely related to $d\vec{T}/ds$.

To describe the bending characteristics of a curve in 3-space completely, you must take into account $d\vec{T}/ds$, $d\vec{N}/ds$, & $d\vec{B}/ds$.

## Curvature

If C is a smooth curve in 2- or 3-space that is parametrized by arc length, then the curvature of C, denoted by $\kappa = \kappa(s)$ is defined by

$$\kappa(s) = \left\| \frac{d\vec{T}}{ds} \right\| = \| \vec{r}\,''(s) \|$$

This formula is only applicable if the curve is parametrized in terms of arc length.

If $\vec{r}(t)$ is a smooth vector-valued fn in 2- or 3-space, then for each value of t at which $\vec{T}\,'(t)$ & $\vec{r}\,''(t)$ exist, the curvature $\kappa$ can be expressed as

$$\kappa(t) = \frac{\| \vec{T}\,'(t) \|}{\| \vec{r}\,'(t) \|} \quad \text{or} \quad \kappa(t) = \frac{\| \vec{r}\,'(t) \times \vec{r}\,''(t) \|}{\| \vec{r}\,'(t) \|^3}$$

If a curve in 2-sp has nonzero curvature $\kappa$ at a point P, then the circle of radius $\rho = \frac{1}{\kappa}$ sharing a common tangent w/ C at P, and centered on the concave side of the curve at P, is called the circle of curvature or osculating circle at P. The osculating circle & the curve C not only touch at P but they have equal curvatures at this point.

The radius $\rho$ of the osculating circle at P is called the radius of curvature at P and the center of the circle is called the center of curvature at P.

A geometric interpretation of curvature in 2-space can be obtained by considering the angle $\phi$ measured CCW from the direction of the positive x-axis to the unit tangent vector $\vec{T}$.

$$\kappa(s) = \left| \frac{d\phi}{ds} \right|$$

which means that curvature in 2-space can be interpreted as the magnitude of the rate of change of $\phi$ wrt to s.

## Formula Summary

$$\vec{T}(s) = \vec{r}\,'(s) \qquad \vec{T}(t) = \frac{\vec{r}\,'(t)}{\| \vec{r}\,'(t) \|}$$

$$\vec{N}(s) = \frac{1}{\kappa(s)} \frac{d\vec{T}}{ds} = \frac{\vec{r}\,''(s)}{\| \vec{r}\,''(s) \|} = \frac{\vec{r}\,''(s)}{\kappa(s)} \qquad \vec{N}(t) = \vec{B}(t) \times \vec{T}(t)$$

$$\vec{B}(s) = \frac{\vec{r}\,'(s) \times \vec{r}\,''(s)}{\| \vec{r}\,''(s) \|} = \frac{\vec{r}\,'(s) \times \vec{r}\,''(s)}{\kappa(s)} \qquad \vec{B}(t) = \frac{\vec{r}\,'(t) \times \vec{r}\,''(t)}{\| \vec{r}\,'(t) \times \vec{r}\,''(t) \|}$$

p.2 13.5

P.895

8) $\vec{r}(t) = t\vec{i} + \frac{1}{2}t^2\vec{j} + \frac{1}{3}t^3\vec{k}$

$\vec{r}'(t) = \vec{i} + t\vec{j} + t^2\vec{k} \quad \vec{r}''(t) = 0\vec{i} + \vec{j} + 2t\vec{k}$

$$\vec{r}'(t)\times\vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & t & t^2 \\ 0 & 1 & 2t \end{vmatrix}$$

$= t^2\vec{i} - 2t\vec{j} + \vec{k}$

$$K(t) = \frac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{\sqrt{t^4+4t^2+1}}{(1+t^2+t^4)^{3/2}}$$

14) $x = \sin t,\ y = \cos t,\ z = \frac{1}{2}t^2;\ t=0$

$\vec{r}' = \cos t\vec{i} - \sin t\vec{j} + t\vec{k}$

$\vec{r}'' = -\sin t\vec{i} - \cos t\vec{j} + \vec{k}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix}$$

$$K(t) = \frac{\sqrt{0+(-1)^2+(-1)^2}}{(\sqrt{1})^3} = \sqrt{2} \qquad \rho = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

26) $x = 4\cos t,\ y = \sin t;\ t = \pi/2$

from 17a

$$K(t) = \frac{|x'y'' - y'x''|}{[(x')^2+(y')^2]^{3/2}}$$

$$K(t) = \frac{|-4\sin t(-\sin t) - \cos t(-4\cos t)|}{[16\sin^2 t + \cos^2 t]^{3/2}}$$

$$K(t) = \frac{|4+0|}{64} = \frac{1}{16}$$

43) $r = \sin 3\theta,\ \theta = 0$

$$K(\theta) = \frac{\left|\sin^2 3\theta + 2(3\cos 3\theta)^2 + 27\sin^2 3\theta\right|}{\left[\sin^2 3\theta + 9\cos^2 3\theta\right]^{3/2}}$$

$$K(0) = \frac{18}{27} = \frac{2}{3}$$

50) $y = x^3,\ x>0$, maximum

$$K(x) = \frac{6x}{(1+(3x^2)^2)^{3/2}} = \frac{6x}{(1+9x^4)^{3/2}}$$

max K:

$$K'(x) = \frac{6(1+9x^4)^{3/2} - \overset{3}{\cancel{6}}x\left[\frac{3}{2}(1+9x^4)^{1/2}(36x^3)\right]}{(1+9x^4)^3}$$

$$K'(x) = \frac{6\cancel{(1+9x^4)^{1/2}}\left[1+9x^4-54x^4\right]}{(1+9x^4)^{\cancel{3}\,5/2}} = \frac{6(1-45x^4)}{(1+9x^4)^{5/2}} \Rightarrow x = 45^{-\frac{1}{4}}$$

60) $y = e^x \quad , \quad y = ax^2+bx+c$

$$K(x) = \frac{|y''|}{[1+(y')^2]^{3/2}}$$

$y_1 = e^x \qquad y_2 = ax^2+bx+c$

$y_1' = e^x \qquad y_2' = 2ax+b$

$y_1'' = e^x \qquad y_2'' = 2a$

@ $x=0 \quad e^0 = c \therefore c=1$

$e^0 = 2a(0)+b \therefore b=1$

$e^0 = 2a \therefore a = \frac{1}{2}$

68) $\vec{r}(t) = a\cos t\vec{i} + a\sin t\vec{j} + ct\vec{k}$

$$\tau(t) = \frac{[\vec{r}'(t)\times\vec{r}''(t)]\cdot\vec{r}'''(t)}{\|\vec{r}'(t)\times\vec{r}''(t)\|^2}$$

$\vec{r}' = -a\sin t\vec{i} + a\cos t\vec{j} + c\vec{k}$

$\vec{r}'' = -a\cos t\vec{i} - a\sin t\vec{j} + 0\vec{k}$

$\vec{r}''' = a\sin t\vec{i} - a\cos t\vec{j} + 0\vec{k}$

$(ac\sin t\vec{i} - ac\cos t\vec{j} + a^2\vec{k})\cdot(a\sin t\vec{i} - a\cos t\vec{j} + 0\vec{k})$

$$= \frac{a^2c\sin^2 t + a^2c\cos^2 t + 0}{(a\sqrt{c^2+a^2})^2}$$

$$\tau(t) = \frac{c}{c^2+a^2}$$

## Projectile Motion

$\vec{F} = m\vec{a}$ $\quad \vec{F} = -mg\vec{j}$

acceleration

$\vec{a} = -g\vec{j}$

velocity

$\vec{v}(t) = -gt\vec{j} + \vec{v}_0$

position

$\vec{r}(t) = (-\frac{1}{2}gt^2 + s_0)\vec{j} + t\vec{v}_0$

(Parametric form)

$\vec{r}(t) = (v_0\cos\theta)t\vec{i} + (s_0 + (v_0\sin\theta)t - \frac{1}{2}gt^2)\vec{j}$

$x = v_0\cos\theta\, t$

$y = s_0 + (v_0\sin\theta)t - \frac{1}{2}gt^2$

$\vec{v}(t) = v_0\cos\theta\,\vec{i} + (v_0\sin\theta - gt)\vec{j}$

$v_x = v_0\cos\theta, \quad v_y = v_0\sin\theta - gt$

## P.908

8) $\vec{r}(t) = e^t\sin t\,\vec{i} + e^t\cos t\,\vec{j} + t\vec{k};\ t = \pi/2$

$\vec{v}(t) = e^t(\sin t + \cos t)\vec{i} + e^t(\cos t - \sin t) + \vec{k}$

$\vec{a}(t) = e^t(\cancel{\sin t} + \cos t + \cos t - \cancel{\sin t})\vec{i} + e^t(\cos t - \sin t - \sin t - \cos t)\vec{j}$

$\vec{v}(\frac{\pi}{2}) = e^{\pi/2}\vec{i} - e^{\pi/2}\vec{j} + \vec{k}$

$\vec{a}(\frac{\pi}{2}) = -2e^{\pi/2}\vec{j}$

18) $\vec{a}(t) = (t+1)^{-2}\vec{j} - e^{-2t}\vec{k};\ \vec{v}(0) = 3\vec{i} - \vec{j},\ \vec{r}(0) = 2\vec{k}$

$d\vec{v} = \int\left(0\vec{i} + (t+1)^{-2}\vec{j} - e^{-2t}\vec{k}\right)dt$

$\vec{v}(t) = \vec{i} - (t+1)^{-1}\vec{j} + \frac{1}{2}e^{-2t}\vec{k} + C$

$3\vec{i} - \vec{j} = \vec{i} - \vec{j} + \frac{1}{2}\vec{k} + C$

$2\vec{i} - \frac{1}{2}\vec{k} = C$

$\vec{v}(t) = 3\vec{i} - (t+1)^{-1}\vec{j} + \frac{1}{2}(e^{-2t} - 1)\vec{k}$

$\frac{d\vec{r}}{dt} = \int\left(3\vec{i} - \frac{1}{(t+1)}\vec{j} + \frac{1}{2}(e^{-2t} - 1)\vec{k}\right)dt$

$\vec{r}(t) = 3t\vec{i} - \ln|t+1|\vec{j} + \frac{1}{2}(-\frac{1}{2}e^{-2t} - t)\vec{k} + C$

$2\vec{k} = -\frac{1}{4}\vec{k} + C$

$\frac{9}{4}\vec{k} = C$

$\vec{r}(t) = 3t\vec{i} - \ln|t+1|\vec{j} - \frac{1}{4}(e^{-2t} + 2t - 9)\vec{k}$

28) $\vec{r} = \cos 2t\,\vec{i} + (1 - \cos 2t)\vec{j} + (3 + \frac{1}{2}\cos 2t)\vec{k}$ $\quad 0 \le t \le \pi$

$\Delta\vec{r} = \vec{r}(\pi) - \vec{r}(0) = (\vec{i} + \frac{7}{2}\vec{k}) - (\vec{i} + \frac{7}{2}\vec{k}) = \vec{0}$

$s = \int_0^\pi \sqrt{(-2\sin 2t)^2 + (2\sin 2t)^2 + (\sin 2t)^2}$

$s = \int_0^\pi |3\sin 2t|\,dt$

$s = 6\int_0^{\pi/2} \sin 2t$

$s = 3\int \sin u\,du$

$s = 3\cos 2t\,|_0^{\pi/2} = 6$

33) $\vec{r} = (t^3 - 2t)\vec{i} + (t^2 - 4)\vec{j};\ t = 1$

$\vec{v} = (3t^2 - 2)\vec{i} + (2t)\vec{j} \Rightarrow \vec{v}(1) = \vec{i} + 2\vec{j}$

$\vec{a} = (6t)\vec{i} + 2\vec{j} \Rightarrow \vec{a}(1) = 6\vec{i} + 2\vec{j}$

a) $a_T = \frac{10}{\sqrt{5}} = 2\sqrt{5}$ $\quad a_N = \frac{\|-10\|}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$

b) $a_T\vec{T} = 2\sqrt{5}\frac{(\vec{i} + 2\vec{j})}{\sqrt{5}} = 2\vec{i} + 4\vec{j}$

$a_N\vec{N} = \vec{a} - a_T\vec{T} = 4\vec{i} - 2\vec{j}$

c) $K = \frac{10}{\sqrt{5}^3} = \frac{10}{5\sqrt{5}} = \frac{2}{\sqrt{5}}$

40) $\vec{v} = 3\vec{i} - 4\vec{k};\ \vec{a} = \vec{i} - \vec{j} + 2\vec{k}$

$\|\vec{v}\| = 5$

$a_T = \frac{-5}{5} = -1$ $\quad a_N = \frac{\|-4\vec{i} - 10\vec{j} - 3\vec{k}\|}{5} = \sqrt{5}$

$\vec{T} = \frac{1}{5}(3\vec{i} - 4\vec{k})$ $\quad \vec{N} = \frac{\vec{a} - a_T\vec{T}}{a_N} = \frac{8\vec{i} - 5\vec{j} + 6\vec{k}}{5\sqrt{5}}$

55) $V_0 = 320$ ft/s; $\theta = 60°$, $s_0 = 0$

a) $x = 320(\frac{1}{2})t = 160t$, $y = 160\sqrt{3}t - 16t^2$

b) $\frac{dy}{dt} = 160\sqrt{3} - 32t \Rightarrow t = 5\sqrt{3}\ \therefore y_{max} = 1200$ ft

c) $y = 0$ @ $t = 0$ & $t = 10\sqrt{3}\ \therefore x_{max} = 1600\sqrt{3}$ ft

d) $\vec{v}(t) = 160\vec{i} + (160\sqrt{3} - 32t)\vec{j}$

$\vec{v}(10\sqrt{3}) = 160\vec{i} + (160\sqrt{3} - 320\sqrt{3})\vec{j} = 160(\vec{i} - \sqrt{3}\vec{j})$

$\|\vec{v}(10\sqrt{3})\| = 320$ ft/s

Assume that the motion of a particle in 2- and 3-space is described by a smooth v.v.f. $\vec{r}(t)$ in which the parameter $t$ is time. We call this the trajectory.

We will define the direction of motion at time $t$ to be the direction of the unit tangent vector $\vec{T}(t)$, and we define the speed to be $ds/dt$; the instantaneous rate of change of the arc length traveled by the article from an arbitrary reference point.

We will combine the speed & direction of motion to form the vector $\vec{v}(t) = \frac{ds}{dt}\vec{T}(t)$ called the velocity.

The instantaneous velocity, acceleration & speed are defined, respectively, by

velocity $\quad \vec{v}(t) = \frac{d\vec{r}}{dt}$

acceleration $\quad \vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$

speed $\quad \|\vec{v}(t)\| = \frac{ds}{dt}$

displacement $\quad \Delta\vec{r} = \int_{t_1}^{t_2} \vec{v}(t)\,dt = \int_{t_1}^{t_2} \frac{d\vec{r}}{dt}\,dt = \vec{r}(t)\Big|_{t_1}^{t_2} = \vec{r}(t_2) - \vec{r}(t_1)$

distance $\quad s = \int_{t_1}^{t_2} \left\|\frac{d\vec{r}}{dt}\right\| dt = \int_{t_1}^{t_2} \|\vec{v}(t)\|\,dt$

Velocity & acceleration vectors

$\vec{v} = \frac{ds}{dt}\vec{T} \qquad \vec{a} = \frac{d^2s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2\vec{N}$ $\quad s$ is arc length, $\vec{T}, \vec{N}, \kappa$ are the unit tangent vector, unit normal vector, and the curvature

The coefficients of $\vec{T}$ & $\vec{N}$ are commonly denoted by

$a_T = \frac{d^2s}{dt^2} \qquad a_N = \kappa\left(\frac{ds}{dt}\right)^2 \quad \therefore \vec{a} = a_T\vec{T} + a_N\vec{N}$

$a_T$ & $a_N$ are called the tangential scalar component of acceleration and the normal scalar component of acceleration. The vectors $a_T\vec{T}$ & $a_N\vec{N}$ are called the tangential vector component of acceleration and the normal vector component of acceleration

The previous formulas for $a_T$ & $a_N$ are very awkward; therefore

$a_T = \frac{\vec{v}\cdot\vec{a}}{\|\vec{v}\|} \qquad a_N = \frac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|} \qquad \kappa = \frac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|^3}$ $\quad$ also, by Pythagorean's Thm

$a_N = \sqrt{\|\vec{a}\|^2 - a_T^2}$