Ch. 16

16.1

A **vector field** is a fn that associates a unique vector $\vec{F}(P)$ with each point P in a region of 2- or 3-space.

Notation: $\vec{F}(x,y)$ or $\vec{F}(x,y,z)$ is vector w/ radius vector $\vec{r} = x\vec{i} + y\vec{j}$ or $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
a vector field can be written as $\vec{F}(\vec{r})$ or simply $\vec{F}$

Inverse Square Field: If $\vec{r}$ is a radius vector in 2- or 3-space, and if c is a constant, then a vector field of the form $\vec{F}(\vec{r}) = \frac{c}{\|\vec{r}\|^3}\vec{r}$ is called an inverse-square field

Gradient Field of ϕ: $\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$

A vector field $\vec{F}$ in 2- or 3-space is said to be conservative in a region if it is the gradient field for some fn ϕ in that region. The fn ϕ is called a potential fn for $\vec{F}$ in the region.

Divergence: If $\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$, then we define the divergence of $\vec{F}$, written div $\vec{F}$, by

$$\text{div}\,\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$$

Curl: If $\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$, then we define the curl of $\vec{F}$, written curl $\vec{F}$, by

$$\text{curl}\,\vec{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\vec{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\vec{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{k}$$

Note: div $\vec{F}$ has scalar values, curl $\vec{F}$ is itself a vector field

del operator: $\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$

The del operator allows div $\vec{F}$ & curl $\vec{F}$ to be expressed in dot product and cross product notation, respectively, as

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

p2 16.1

Laplacian Operator: $\vec{\nabla}^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Laplace Eqn: $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z} = 0$

P. 1099

12b) $\phi(x,y,z) = x\sin z + y\sin x + z\sin y$
$\vec{F}(x,y,z) = (\sin z + y\cos x)\vec{i} + (\sin x + z\cos y)\vec{j} + (\sin y + x\cos z)\vec{k}$

$\vec{\nabla}\phi = (\sin z + y\cos x)\vec{i} + (\sin x + z\cos y)\vec{j} + (x\cos z + \sin y)\vec{k} = \vec{F}$

14) $\vec{F}(x,y,z) = xz^3\vec{i} + 2y^4x^2\vec{j} + 5z^2y\vec{k}$

$\text{div}\,\vec{F} = z^3 + 8y^3x^2 + 10zy$ $\quad$ $\text{Curl}\,\vec{F} = (5z^2 - 0)\vec{i} + (3xz^3 - 0)\vec{j} + (4y^4x - 0)\vec{k}$
$= (5z^2)\vec{i} + (3xz^3)\vec{j} + (4y^4x)\vec{k}$

19) $\vec{F}(x,y,z) = 2x\vec{i} + \vec{j} + 4y\vec{k}$
$\vec{G}(x,y,z) = x\vec{i} + y\vec{j} - z\vec{k}$

$\vec{F}\times\vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2x & 1 & 4y \\ x & y & -z \end{vmatrix} = -(z + 4y^2)\vec{i} + (2xz + 4xy)\vec{j} + (2xy - x)\vec{k}$

$\vec{\nabla}\cdot[-(z+4y^2)\vec{i} + (2xz + 4xy)\vec{j} + (2xy - x)\vec{k}] = 0 + 4x + 0 = 4x$

23) $\vec{F}(x,y,z) = 0\vec{i} + xy\vec{j} + xyz\vec{k}$ $\quad$ $\vec{\nabla}\times(\vec{\nabla}\times\vec{F})$

$\text{Curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xy & xyz \end{vmatrix} = \left(\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}(xy)\right)\vec{i} - \left(\frac{\partial}{\partial x}xyz - 0\right)\vec{j} + \left(\frac{\partial}{\partial x}xy - 0\right)\vec{k}$

$\vec{\nabla}\times[xz\vec{i} - yz\vec{j} + y\vec{k}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -yz & y \end{vmatrix} = \left(\frac{\partial}{\partial y}y + \frac{\partial}{\partial z}yz\right)\vec{i} - \left(0 - \frac{\partial}{\partial z}xz\right)\vec{j} + (0 - 0)\vec{k}$
$= (1 + y)\vec{i} + x\vec{j} + 0k$

43a) $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$
slope $= -\frac{x}{y}$; the slope of the tangent line to C @ $(x,y) = \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = -\frac{x}{y}$

b) $y\,dy = -x\,dx \quad \int \quad x^2 + y^2 = C$
$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$

16.2

In 2-space:

The line integral of f wrt s along C: $\int_C f(x,y)\, ds$ where s is arc length

with this, surface area becomes $A = \int_C f(x,y)\, ds$

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad \text{in parametric form}$$

In 3-space:

The line integral of f wrt s along C: $\int_C f(x,y,z)\, dS$

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \quad \text{in parametric form}$$

Mass $M = \int_C \delta(x,y)\, ds$ or $M = \int_C \delta(x,y,z)\, ds$

If C is a smooth parametric curve in 2- or 3-space, then its arc length L can be expressed as

$$L = \int_C ds$$

line integral wrt x, y, z:

$$\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x,y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

$$\int_C f(x,y,z)\, dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\, dt$$

$$\int_C f(x,y,z)\, dy = \int_a^b f(x(t), y(t), z(t))\, y'(t)\, dt$$

$$\int_C f(x,y,z)\, dz = \int_a^b f(x(t), y(t), z(t))\, z'(t)\, dt$$

$$\int_C f(x,y)\, dx + g(x,y)\, dy = \int_C f(x,y)\, dx + \int_C g(x,y)\, dy$$

$$\int_C f(x,y,z)\, dx + g(x,y,z)\, dy + h(x,y,z)\, dz$$

$$= \int_C f(x,y,z)\, dx + \int_C g(x,y,z)\, dy + \int_C h(x,y,z)\, dz$$

The notion of the line integral can be extended to curves formed from finitely many smooth curves $C_1, C_2, \ldots, C_n$ joined end to end. Such a curve is called piecewise smooth

<u>Independence of Parametrization</u> - the value of the line integral along a curve C does not depend on the parametrization of C in the sense that any two parametrizations of C w/ the same orientation produce the same value for the line integral

<u>Reversal of Orientation</u> - If C is a smooth curve, then a smooth change of parameter that reverses the orientation of C changes the sign of a line integral along C wrt x, y, or z, but leave the value of a line integral along C wrt arc length unchanged.

If $\vec{F}$ is a continuous vector and C is a smooth parametric curve in 2- or 3-space w/ unit tangent vector $\vec{T}$, then the work performed by the vector field on a particle that moves along C in the direction of increasing parameter is

$\vec{W} = \int_C \vec{F}\cdot\vec{T}\,ds$ since $\vec{T} = \frac{d\vec{r}}{ds}$, $\vec{W} = \int_C \vec{F}\cdot d\vec{r}$

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$ or $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Work in scalar form:

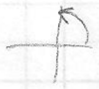
$W = \int_C f(x,y)\,dx + g(x,y)\,dy$ or $W = \int_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$

<u>P. 1112</u>

10) $\int_C \frac{e^{-z}}{x^2+y^2}\,ds$ C: $x = 2\cos t$, $y = 2\sin t$, $z = t$, $(0 \le t \le 2\pi)$

$\int_0^{2\pi} \frac{e^{-t}}{4}\sqrt{4\sin^2 t + 4\cos^2 t + 1}\,dt \Rightarrow \frac{\sqrt{5}}{4}\int_0^{2\pi} e^{-t}\,dt = \frac{\sqrt{5}}{4}\left[-e^{-2\pi}+1\right] = \frac{\sqrt{5}}{4}\left[1-e^{-2\pi}\right]$

15) $\int_C (x^2+y^2)\,dx - x\,dy$; C: $x^2+y^2=1$ ccw $(1,0)\to(0,1)$



$x = \cos t$, $y = \sin t$; $0 \le t \le \pi/2$

$\int_C (x^2+y^2)\,dx - \int_C x\,dy$

$\int_0^{\pi/2} \left[(-\sin t) - \cos^2 t\right]dt \Rightarrow \int_0^{\pi/2}\left[-\sin t - \frac{1}{2}(1+\cos 2t)\right]dt$

$\Rightarrow \left[\cos t - \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right)\right]\Big|_0^{\pi/2}$

$\left(0 - \frac{1}{2}\left(\frac{\pi}{2}+0\right)\right) - \left(1 - \frac{1}{2}(0+0)\right)$

$-1 - \frac{\pi}{4}$

p.3 16.2

P. 1112

29) $y=\sqrt{9-x^2}\ (0\le x\le 3)$; $\delta(x,y)=x\sqrt{y}$

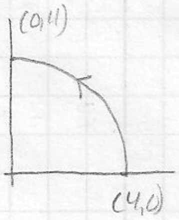
$m=\int_C \delta(x,y)\,ds$

$m=\int x\sqrt{y}\ ds$; $x=3\cos t,\ y=3\sin t$; $0\le t\le \pi/2$

$m=\int_0^{\pi/2} 3\cos t\sqrt{3\sin t}\ \sqrt{(-3\sin t)^2+(3\cos t)^2}\ dt$

$m=9\sqrt{3}\int_0^{\pi/2}\cos t\sqrt{\sin t}\ dt = \overset{3}{\cancel{9}}\sqrt{3}\cdot\frac{2}{\cancel{3}}(\sin t)^{3/2}\Big|_0^{\pi/2}=\underline{\underline{6\sqrt{3}}}$

39)



$\vec{F}(x,y)=\frac{1}{x^2+y^2}\,\vec{i}+\frac{4}{x^2+y^2}\,\vec{j}$

$W=\int_0^{\pi/2}\vec{F}\cdot d\vec{r}$

$x=4\cos t,\ y=4\sin t$; $W=\int_0^{\pi/2}\left(\frac{1}{16}\vec{i}+\frac{1}{4}\vec{j}\right)\cdot\left(-4\sin t\,\vec{i}+4\cos t\,\vec{j}\right)dt$

$W=\int_0^{\pi/2}\left(-\frac{1}{4}\sin t+\cos t\right)dt$

$W=\left[\frac{1}{4}\cos t+\sin t\right]\Big|_0^{\pi/2}=1-\frac{1}{4}=\underline{\underline{3/4}}$

41) $y=x^2,\ 0\le x\le 2$; $z=3x$ $\Rightarrow x=t,\ y=t^2,\ 0\le t\le 2$

$A=\int_C f(x,y)\,ds=\int_0^2 3t\sqrt{(1)^2+(2t)^2}\ dt$

$=3\int_0^2 t\sqrt{1+4t^2}\ dt$ $\quad u=1+4t^2$, $du=8t\,dt$

$=\frac{3}{8}\int u^{1/2}\,du$

$=\frac{3}{8}\cdot\frac{2}{3}(1+4t^2)^{3/2}\Big|_0^2$

$=\underline{\underline{\frac{1}{4}\left(17^{3/2}-1\right)}}$

From 16.2, work is $W = \int_C \vec{F} \cdot d\vec{r}$. The parametric curve C in a work integral is called the path of integration

Fundamental Thm of Work Integral - Suppose $\vec{F}(x,y) = f(x,y)\hat{i} + g(x,y)\hat{j}$ is a conservative vector field in some open region D containing the point (x_0, y_0) and (x_1, y_1) and that f & g are continuous in that region. If $\vec{F}(x,y) = \vec{\nabla}\phi(x,y)$ and if C is any piecewise smooth parametric curve that starts at (x_0, y_0), ends at (x_1, y_1), and lies in the region D, then

$$\int_C \vec{F}(x,y) \cdot d\vec{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$

or, equivalently

$$\int_C \vec{\nabla}\phi \cdot d\vec{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$

This shows that the value of a work integral along a piecewise smooth path in a conservative vector field is independent of path.

If $f(x,y)$ & $g(x,y)$ are continuous on some open connected region D, then the following statements are equivalent (all true or all false):

(a) $\vec{F}(x,y) = f(x,y)\hat{i} + g(x,y)\hat{j}$ is a conservative vector field on the region D.

(b) $\int_C \vec{F} \cdot d\vec{r} = 0$ $\forall$ piecewise smooth closed curve C in D

(c) $\int_C \vec{F} \cdot d\vec{r}$ is independent of the path from any point P in D to any point Q in D $\forall$ piecewise smooth curve C in D.

Conservative Field Test - If $f(x,y)$ & $g(x,y)$ are continuous and have continuous first partial derivatives on some open region D, and if the vector field $\vec{F}(x,y) = f(x,y)\hat{i} + g(x,y)\hat{j}$ is conservative on D, then

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad \text{as each point in } D.$$

Conversely, if D is simply connected and $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ holds at each point in D, then $\vec{F}(x,y) = f(x,y)\hat{i} + g(x,y)\hat{j}$ is conservative.

In 3-space, $\int_C \vec{F}(x,y,z) \cdot d\vec{r} = \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0)$

Conservative field,

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y} \quad \text{; that is curl } \vec{F} = 0$$

p.2 16.3

Conservation of Energy - If $\vec{F}(x,y,z)$ is a conservative force field w/ a potential fn $\phi(x,y,z)$, then $V(x,y,z) = -\phi(x,y,z)$ is the potential energy of the field at the point (x,y,z).

Conservation of energy principle. $\frac{1}{2}mv_f^2 + V_f = \frac{1}{2}mv_i^2 + V_i$

The total energy of the particle (kinetic + potential) does not change as the particle moves along a path in a conservative vector field.

P. 1122

6) $\vec{F}(x,y) = x\ln y\,\vec{i} + y\ln x\,\vec{j}$

$\frac{\partial f}{\partial y} = \frac{x}{y}$ $\frac{\partial g}{\partial x} = \frac{y}{x}$ not conservative

14) $\int_{(1,1)}^{(3,3)} \left(e^x\ln y - \frac{e^y}{x}\right)dx + \left(\frac{e^x}{y} - e^y\ln x\right)dy$; x & y both positive

show $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, then is conservative; by Thm 16.3.2, if $\vec{F}$ is conservative then $\int_C \vec{F}\cdot d\vec{r}$ is independent.

$\frac{\partial f}{\partial y} = \frac{e^x}{y} - \frac{e^y}{x}$; $\frac{\partial g}{\partial x} = \frac{e^x}{y} - \frac{e^y}{x}$ ∴ independent

so $\phi(x_1,y_1) - \phi(x_0,y_0)$

$\phi = e^x\ln y - e^y\ln x \Rightarrow \phi(3,3) - \phi(1,1) = (e^3\ln 3 - e^3\ln 3) - (e^1\ln 1 - e^1\ln 1) = \underline{\underline{0}}$

16) $\vec{F}(x,y) = 2xy^3\,\vec{i} + 3x^2y^2\,\vec{j}$; $P(-3,0)$, $Q(4,1)$

$\frac{\partial f}{\partial y} = 6xy^2 = \frac{\partial g}{\partial x}$ ∴ conservative

$W = \int_C \vec{F}\cdot d\vec{r} = \int_{(-3,0)}^{(4,1)} (2xy^3\,dx) + (3x^2y^2)\,dy = \phi(4,1) - \phi(-3,0)$

to find ϕ

$\phi(x,y) = x^2y^3 + k(y)$

$\frac{\partial\phi}{\partial y} = 3x^2y^2 + k'(y) = 3x^2y^2 \Rightarrow k'(y) = 0$

$k(y) = K$

∴ $\phi(x,y) = x^2y^2 + K$

$W = [4^2 1^2 + K] - [(-3)^2(0)^2 + K]$

$W = 16 + 0 = \underline{\underline{16}}$

P.1122

19) $\vec{F}(x,y) = (e^y + ye^x)\vec{i} + (xe^y + e^x)\vec{j}$

C: $\vec{r}(t) = (\sin \frac{\pi t}{2})\vec{i} + (\ln t)\vec{j}$ $(1 \le t \le 2)$

$\frac{\partial f}{\partial y} = e^y + e^x = \frac{\partial g}{\partial x}$ ∴ conservative

then $\phi(x,y) = xe^y + ye^x + k(y)$

$\frac{\partial \phi}{\partial y} = xe^y + e^x + k'(y) = xe^y + e^x \Rightarrow k'(y) = 0 \Rightarrow k(y) = k$

∴ $\phi(x,y) = xe^y + ye^x + k$

$\phi(0, \ln 2) - \phi(1,0) = [0 + \ln 2 + k] - [1 + 0 + k] = \underline{\underline{\ln 2 - 1}}$

25) If $\vec{F}$ is conservative, then $\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$

$\Rightarrow f = \frac{\partial \phi}{\partial x}$, $g = \frac{\partial \phi}{\partial y}$, $h = \frac{\partial \phi}{\partial z}$

$\frac{\partial f}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ & $\frac{\partial g}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$; $\frac{\partial f}{\partial z} = \frac{\partial^2 \phi}{\partial z \partial x}$ & $\frac{\partial h}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial z}$; $\frac{\partial g}{\partial z} = \frac{\partial^2 \phi}{\partial z \partial y}$ & $\frac{\partial h}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial z}$

$\Rightarrow \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$, $\frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial^2 \phi}{\partial x \partial z}$, $\frac{\partial^2 \phi}{\partial z \partial y} = \frac{\partial^2 \phi}{\partial y \partial z}$ from Thm 14.3.2 (p.948)

equality of mixed second partials

Green's Thm - Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve C oriented ccw. If $f(x,y)$ & $g(x,y)$ are continuous and have continuous first partial derivatives on some open set containing R, then

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \qquad (1)$$

It is common practice to denote a line integral around a simple closed curve by

$$\oint_C f(x,y)\,dx + g(x,y)\,dy$$

Sometimes a direction arrow is added to the circle to indicate whether the integration is CW or CCW.

$$\oint_C f(x,y)\,dx + g(x,y)\,dy$$

The following 3 formulas express the area A of a region R in terms of line integrals around the boundary:

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C -y\,dx + x\,dy$$

Although the third form looks more complicated, it often leads to simpler integration; but each has advantages in certain situations

P.1129

6) $\oint_C y\tan^2 x\,dx + \tan x\,dy$, $C: x^2+(y+1)^2=1$

$(0,\pi)$

$x^2+y^2+2y+1=1$
$r^2+2r\sin\theta=0$
$r(r+2\sin\theta)=0$
$r=0,\ 2\sin\theta$

$$\int\int (\sec^2 x - \tan^2 x)\,dy\,dx$$

$$\int\int 1\,dy\,dx$$

$$\int_0^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta$$

$$\int_0^{\pi} \frac{1}{2}r^2\Big|_0^{2\sin\theta}\,d\theta$$

$$\int_0^{\pi} 2\sin^2\theta\,d\theta$$

$$\int_0^{\pi} \frac{1}{\cancel{2}}\cancel{2}(1-\cos 2\theta)\,d\theta$$

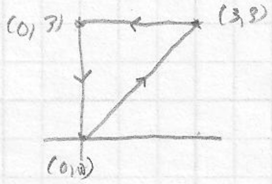
$$\int_0^{\pi}(1-\cos 2\theta)\,d\theta$$

$$\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi}$$

$$(\pi - 0) - (0 - 0)$$

$$\underline{\underline{\pi}}$$

12) $\oint_C \cos x \sin y\,dx + \sin x \cos y\,dy$; C: triangle w/ vertices $(0,0), (3,3), (0,3)$



$$\int\int (\cos x \cos y - \cos x \cos y)\,dy\,dx$$

$$\int_0^3\int_0^3 0\,dy\,dx = \underline{0}$$

24) Use Green's Thm to prove the centroid $(\bar{x}, \bar{y})$ of R is given by

$$\bar{x} = \frac{1}{2A}\oint_C x^2\,dy \quad , \quad \bar{y} = -\frac{1}{2A}\oint_C y^2\,dx$$

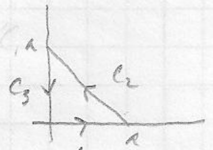
$\bar{x} = \frac{1}{A}\int\int_R x\,dA$ from Green's Thm $\iint_R x\,dA = \frac{1}{2}\int x^2\,dy$

$\therefore \bar{x} = \frac{1}{A}\left[\frac{1}{2}\int x^2\,dy\right] = \frac{1}{2A}\oint_C x^2\,dy$

$\bar{y} = \frac{1}{A}\int_R\int y\,dA$ from Green's Thm $\iint_R y\,dA = \frac{1}{2}\int y^2\,dx$

$\therefore \bar{y} = \frac{1}{A}\left[\frac{1}{2}\int y^2\,dx\right] = \frac{1}{2A}\oint_C \frac{1}{2}y^2\,dx$

26) $A = \frac{1}{2}bh = \frac{a^2}{2}$



$C_1: x = t, y = 0, 0 \le t \le a$

$C_2: x = a-t, y = t, 0 \le t \le a$

$C_3: x = 0, y = a-t, 0 \le t \le a$

$\bar{x} = \frac{1}{2A}\oint_C x^2\,dy \quad , \quad \bar{y} = -\frac{1}{2A}\oint_C y^2\,dx$

$\bar{x} = \frac{1}{2\left(\frac{a^2}{2}\right)}\oint_{C_1} x^2\,dy = \frac{1}{a^2}\oint_{C_1} x^2(0) = 0$

$\bar{x} = \frac{1}{a^2}\oint_{C_2} x^2\,dy = \frac{1}{a^2}\int_0^a (a-t)^2\,dt = -\frac{1}{3a^2}(a-t)^3\Big|_0^a = \frac{1}{3}a$

$\bar{x} = \frac{1}{a^2}\oint_{C_3} x^2\,dy = \frac{1}{a^2}\int_{C_3} x^2\,dy = 0 \quad \therefore \bar{x} = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + \frac{1}{3}a + 0 = \underline{\frac{1}{3}a}$

$\bar{y} = -\frac{1}{2A}\oint_C y^2\,dx = 0 - \int_0^a t^2\,dt + 0 = -\frac{1}{3}a^3 \Rightarrow -\frac{1}{a^2}\left(-\frac{1}{3}a^3\right) \Rightarrow \bar{y} = \frac{1}{3}a \quad \therefore \underline{\left(\frac{a}{3}, \frac{a}{3}\right)}$

Thm 16.5.1 Let σ be a smooth parametric surface whose vector eq'n is

$\vec{r} = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$

where (u,v) varies over a region R in the uv-plane. If $f(x,y,z)$ is continuous on σ, then

$$\iint_\sigma f(x,y,z)\,dS = \iint_R f(x(u,v), y(u,v), z(u,v)) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| dA$$

Thm 16.5.2

a) Let σ be a surface w/ eq'n $z = g(x,y)$ and let R be its projection on the xy-plane. If g has continuous first partials on R and $f(x,y,z)$ is continuous on σ, then

$$\iint_\sigma f(x,y,z)\,dS = \iint_R f(x,y,g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA$$

b) If σ is $y = g(x,z)$ & R is in the xz plane (same conditions as above)

$$\iint_\sigma f(x,y,z)\,dS = \iint_R f(x, g(x,z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\; dA$$

c) If σ is $x = g(y,z)$ & R is in the yz-plane (same conditions as in a))

$$\iint_\sigma f(x,y,z)\,dS = \iint_R f(g(y,z), y, z) \sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\; dA$$

A curved lamina is an idealized object that is thin enough to be viewed as a surface in 3-space.

If the composition of a curved lamina is uniform so that its mass density is distributed uniformly, it is said to be homogeneous.

The mass density is the total mass divided by the total surface area.

If the mass is not uniformly distributed, then this is not a useful measure. In this case, we describe the mass concentration at a point by a <u>mass density fn δ</u>

$$M = \iint_\sigma \delta(x,y,z)\,dS$$

Thm 16.5.3 If σ is a smooth parametric surface in 3-space, then its surface area S can be expressed as

$$S = \iint_\sigma dS$$

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4) $f(x,y,z) = (x^2+y^2)z$; σ : $x^2+y^2+z^2=4$ above $z=1$

$z = \sqrt{4-(x^2+y^2)}$ w/in the circle $x^2+y^2+(1)^2=4$

$x^2+y^2 = 3$

$$\iint_\sigma (x^2+y^2)z\, dS = \iint_R (x^2+y^2)\sqrt{4-(x^2+y^2)}\sqrt{\frac{x^2}{4-(x^2+y^2)} + \frac{y^2}{4-(x^2+y^2)} + 1}\, dA$$

$$= \iint_R 2(x^2+y^2)\, dA = 2\int_0^{2\pi}\int_0^{\sqrt{3}} r^2 (r\, dr\, d\theta)$$

$$= 2\int_0^{2\pi}\int_0^{\sqrt{3}} r^3\, dr\, d\theta$$

$$= 2\int_0^{2\pi} \frac{1}{4}r^4 \Big|_0^{\sqrt{3}} d\theta = \frac{9}{2}\int_0^{2\pi} d\theta = \underline{\underline{9\pi}}$$

11) $\iint_\sigma xyz\, dS$; σ: $2x+3y+4z=12$ in first octant

xy-plane

$$\iint_\sigma xyz\, dS = \iint_R xy\left[\frac{12-2x-3y}{4}\right]\sqrt{\frac{1}{4}+\frac{9}{16}+1}\, dA$$

$$= \iint_R xy\left[\frac{12-2x-3y}{4}\right]\sqrt{\frac{29}{16}}\, dA \quad \leftarrow$$

$$= \frac{\sqrt{29}}{16}\int_0^6\int_0^{(12-2x)/3} xy(12-2x-3y)\, dA$$

$$= \frac{\sqrt{29}}{16}\int_0^6\int_0^{(12-2x)/3} (12xy - 2x^2y - 3xy^2)\, dy\, dx$$

$$= \frac{\sqrt{29}}{16}\int_0^6 (6xy^2 - x^2y^2 - xy^3)\Big|_0^{(12-2x)/3} dx$$

$$= \frac{\sqrt{29}}{16}\int_0^6 \left[\frac{2}{3}x(12-2x)^2 - \frac{x^2(12-2x)^2}{9} - \frac{x(12-2x)^3}{27}\right] dx$$

$$= \frac{\sqrt{29}}{16}\int_0^6 \frac{x(12-2x)^2}{27}\left[18-3x-(12-2x)\right] dx$$

$$= \frac{\sqrt{29}}{432}\int_0^6 x(12-2x)^2(6-x)\, dx = \frac{\sqrt{29}}{432}\int_0^6 4x(6-x)^3\, dx$$

$$= \frac{\sqrt{29}}{108}\int_0^6 (216x - 108x^2 + 18x^3 - x^4)\, dx$$

$$= \frac{\sqrt{29}}{108}\left[108x^2 - 36x^3 + \frac{9}{2}x^4 - \frac{1}{5}x^5\right]_0^6 = \frac{\sqrt{29}}{108}\left[108(6)^2 - 36(6)^3 + \frac{9}{2}(6)^4 - \frac{1}{5}(6)^5\right]$$

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11) xz-plane

$$\iint_\sigma xyz\, dS = \iint_R xz\left(\frac{12-2x-4z}{3}\right)\sqrt{\frac{4}{9}+\frac{16}{9}+1}\; dA$$

$$= \frac{\sqrt{29}}{9}\int_0^3\int_0^{6-2z} xz(12-2x-4z)\, dx\, dz$$

21) $y^2 = 4-z$, between $x=0, x=3, y=0, y=3$; $\delta(x,y,z)=y$

$z = 4-y^2 \quad \frac{\partial z}{\partial x} = 0 \quad \frac{\partial z}{\partial y} = -2y$

$M = \int\int \delta(x,y,z)\, dS$

$M = \int_0^3\int_0^3 y\sqrt{4y^2+1}\; dy\, dx$

$M = \frac{1}{8}\int_0^3 \frac{2}{3}(4y^2+1)^{3/2}\Big|_0^3\; dx$

$M = \frac{1}{12}\int_0^3 (37^{3/2}-1)\, dx$

$M = \frac{(37\sqrt{37}-1)}{12}\, x\Big|_0^3 = \frac{(37\sqrt{37}-1)}{4}$

30) $f(x,y,z) = e^{-z}$; $\vec{r}(u,v) = 2\sin u\cos v\,\vec{i} + 2\sin u\sin v\,\vec{j} + 2\cos u\,\vec{k}$
$(0 \le u \le \pi/2)$, $(0 \le v \le 2\pi)$

$\frac{\partial\vec{r}}{\partial u} = 2\cos u\cos v\,\vec{i} + 2\cos u\sin v\,\vec{j} - 2\sin u\,\vec{k}$

$\frac{\partial\vec{r}}{\partial v} = -2\sin u\sin v\,\vec{i} + 2\sin u\cos v\,\vec{j} + 0\vec{k}$

$$\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\cos u\cos v & 2\cos u\sin v & -2\sin u \\ -2\sin u\sin v & 2\sin u\cos v & 0 \end{vmatrix}$$

$= 4\sin^2 u\cos v\,\vec{i} - 4\sin^2 u\sin v\,\vec{j} + (4\sin u\cos u\cos^2 v + 4\sin u\cos u\sin^2 v)\vec{k}$

$= 4\sin^2 u\cos v\,\vec{i} - 4\sin^2 u\sin v\,\vec{j} + 4\sin u\cos u\,\vec{k}$

$\left\|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right\| = \sqrt{16\sin^4 u\cos^2 v + 16\sin^4 u\sin^2 v + 16\sin^2 u\cos^2 u}$

$= \sqrt{16\sin^4 u + 16\sin^2 u\cos^2 u} = 4\sin u$

e^4 16.5
(cont'd)

30) $= \int_0^{2\pi}\int_0^{\pi/2} e^{-2\cos u}\, 4\sin u \, du\, dv$

$= 4\int_0^{2\pi}\int_0^{\pi/2} \sin u \, e^{-2\cos u}\, du\, dv$ $w = -2\cos u$
$dw = 2\sin u \, du$

$= 2\int_0^{2\pi} e^{-2\cos u} \Big|_0^{\pi/2} dv$

$= 2(1-e^{-2})\int_0^{2\pi} dv = \underline{\underline{4\pi(1-e^{-2})}}$

A two-sided surface is orientable

A one-sided surface is nonorientable

Vectors $\vec{n}$ & $-\vec{n}$ point to opposite sides of the surface.

(from 15.4) A smooth parametric surface σ is given by the vector eq'n

$$\vec{r} = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

then the unit normal

$$\vec{n} = \vec{n}(u,v) = \frac{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|}{\left\|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right\|}$$

$\vec{n}$ points in the positive direction
$-\vec{n}$ points in the negative direction of the surface

Fluid - used to describe both liquids & gases

Liquids are regarded to be incompressible, they have uniform density

Gases are regarded to be compressible, the density may vary from point to point

<u>The flux of $\vec{F}$ across σ</u> $\quad \Phi = \iint_\sigma \vec{F}(x,y,z) \cdot \vec{n}(x,y,z)\, dS$

positive flux - in one unit of time a greater volume of fluid passes through σ in the positive direction than in the negative direction

negative flux - in one unit of time a greater volume of fluid passes through σ in the negative direction than in the positive direction

zero flux - same volume passes σ in each direction

If the fluid has mass density δ, then $\Phi\delta$ (volume x density) represents the net mass of fluid that passes through σ per unit time.

Thm 16.6.2

Let σ be a smooth parametric surface rep. by the vector eqn $\vec{r} = \vec{r}(u,v)$ in which (u,v) varies over a region R in the uv plane. If the component fns of the vector field $\vec{F}$ are continuous on σ, if $\vec{n}$ determines the positive orientation of σ, then

$$\Phi = \int_\sigma\int \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}\cdot\left(\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right) dA$$ [the rhs integrand is in terms of u & v]

from table 16.6.1 $\vec{n} = \dfrac{\vec{\nabla} G}{\|\nabla G\|}$ $\quad \vec{\nabla} G = \dfrac{\partial \vec{r}}{\partial u}\times\dfrac{\partial \vec{r}}{\partial v}$; * if $G = z - g(x,y)$

$$\vec{\nabla} G = -\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k} = -\frac{\partial z}{\partial x}\vec{i} - \frac{\partial z}{\partial y}\vec{j} + \vec{k}$$

$\therefore$ the integral above may be expressed as

$$\Phi = \iint_\sigma \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}\cdot\vec{\nabla} G\, dA$$ (see thm 16.6.3)

This can either be used for direct computation or to derive more specific formulas for each surface type (see table 16.6.1)

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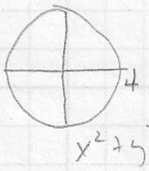
6) $x = r\cos\theta,\ y = r\sin\theta,\ z = r$; $0 \le r \le 1,\ 0 \le \theta \le 2\pi$

$$\vec{n} = \frac{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}}{\|\quad\|} = \frac{-r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\vec{k}}{1} = -r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\vec{k}$$

b) inward

10) $\vec{F} = 0\vec{i} + y\vec{j} + \vec{k}$; σ: paraboloid $z = x^2+y^2$ below $z=4$; $\downarrow \vec{n}$

$$\int\int \vec{F}\cdot\vec{n}\, dS = \int\int (0\vec{i} + y\vec{j} + \vec{k})\cdot(2x\vec{i} + 2y\vec{j} - \vec{k})\, dA$$



$x^2+y^2=4$

$$= \int\int (2y^2 - 1)\, dA = \int_0^{2\pi}\int_0^2 (2r^2\sin^2\theta - 1)\, r\,dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (2r^3\sin^2\theta - r)\, dr\, d\theta = \int_0^{2\pi}\left[\frac{1}{2}r^4\sin^2\theta - \frac{1}{2}r^2\right]_0^2 d\theta$$

$$= \int_0^{2\pi}[8\sin^2\theta - 2]\, d\theta = \int_0^{2\pi}\left[4(1-\cos 2\theta) - 2\right] d\theta = 4\left[\theta - \frac{1}{2}\sin 2\theta - 2\theta\right]_0^{2\pi} = \underline{\underline{4\pi}}$$

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14) $\vec{F} = e^{-y}\hat{i} - y\hat{j} + x\sin z\hat{k}$; σ: $\vec{r}(u,v) = 2\cos v\,\hat{i} + \sin v\,\hat{j} + u\hat{k}$, $0 \le u \le 5$, $0 \le v \le 2\pi$

$\frac{\partial \vec{r}}{\partial u} = 0\hat{i} + 0\hat{j} + 1\hat{k}$

$\frac{\partial \vec{r}}{\partial v} = -2\sin v\,\hat{i} + \cos v\,\hat{j} + 0\hat{k}$. $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = -\cos v\,\hat{i} - 2\sin v\,\hat{j} + 0\hat{k}$

$$\int\int \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) dA = \int\int \left(e^{-\sin v}\hat{i} - \sin v\,\hat{j} + 2\cos v \sin u\,\hat{k}\right) \cdot \left(-\cos v\,\hat{i} - 2\sin v\,\hat{j} + 0\hat{k}\right) dA$$

$$\int\int \left(-\cos v\, e^{-\sin v} + 2\sin^2 v\right) dA$$

$$\int_0^{2\pi} \int_0^5 \left[2\sin^2 v - \cos v\, e^{-\sin v}\right] du\, dv$$

$$5\int_0^{2\pi} \left[(1 - \cos 2v) - \cos v\, e^{-\sin v}\right] dv = 5\left[v - \frac{1}{2}\sin 2v + e^{-\sin v}\right]_0^{2\pi}$$

$$= 5\left[2\pi - 0 + e^0 - 0 - 0 - e^0\right] = \underline{\underline{10\pi}}$$

22) $\vec{F} = -y\hat{i} + z\hat{j} + 3x\hat{k}$

a) net volum upward throug $z = \sqrt{9 - x^2 - y^2}$ in 1s

$$~~\Phi = \int_R\int \left(-y\hat{i} + z\hat{j} + 3x\hat{k}\right) \cdot \left(\frac{-x}{\sqrt{9-x^2-y^2}}\hat{i} + \frac{y}{\sqrt{9-x^2-y^2}} + k\right) dA~~$$

$$~~\Phi = \int\int \left(\frac{-xy}{\sqrt{9-x^2-y^2}} + \frac{y\sqrt{9-x^2-y^2}}{\sqrt{9-x^2-y^2}} + 3x\right) dA~~$$

Thats awful. Since we have a hemisphere, maybe spherical

(p.1141)

$\vec{r}(\phi, \theta) = a\sin\phi\cos\theta\,\hat{i} + a\sin\phi\sin\theta\,\hat{j} + a\cos\phi\,\hat{k}$

$\vec{r}(\phi, \theta) = 3\sin\phi\cos\theta\,\hat{i} + 3\sin\phi\sin\theta\,\hat{j} + 3\cos\phi\,\hat{k} \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\cos\phi\cos\theta & 3\cos\phi\sin\theta & -3\sin\phi \\ -3\sin\phi\sin\theta & 3\sin\phi\cos\theta & 0 \end{vmatrix}$

$\vec{F} = -y\hat{i} + z\hat{j} + 3x\hat{k} = -3\sin\phi\sin\theta\,\hat{i} + 3\cos\phi\,\hat{j} + 9\sin\phi\cos\theta\,\hat{k}$

$$\Phi = \int_R\int \vec{F} \cdot \vec{n}\, dS = \int\int \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}\right) dA$$

$$= \int_R\int \left(-3\sin\phi\sin\theta\,\hat{i} + 3\cos\phi\,\hat{j} + 9\sin\phi\cos\theta\,\hat{k}\right) \cdot \left(9\sin^2\phi\cos\theta\,\hat{i} + 9\sin^2\phi\sin\theta\,\hat{j} + 9\sin\phi\cos\phi\,\hat{k}\right) dA$$

$$\Phi = \int_R\int \left(-27\sin^3\phi\sin\theta\cos\theta + 27\sin^2\phi\cos\phi\sin\theta + 81\sin^2\phi\cos\phi\cos\theta\right) dA$$

$$\Phi = \int_R\int 9\sin\phi\left[-3\sin^2\phi\cos\theta\sin\theta + 3\sin\phi\cos\phi\sin\theta + 9\sin\phi\cos\phi\cos\theta\right] dA$$

The Divergence Thm or Gauss's Thm provide us w/ a physical interpretation of divergence in the context of fluid flow.

Divergence Thm - Let G be a solid whose surface σ is oriented outward. If

$\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$

where f, g, & h have continuous first partial on some open set containing G, and if $\vec{n}$ is the outward unit normal on σ, then

$$\iint_\sigma \vec{F}\cdot\vec{n}\, dS = \iiint_G \text{div}\,\vec{F}\, dV$$

The flux of a vector field across a closed surface w/ outward orientation is equal to the triple integral of the divergence over the region enclosed by the surface. This is sometime called the outward flux across the surface

The outward flux density of $\vec{F}$ @ P_0:

$$\text{div}\,\vec{F}(P_0) = \lim_{vol(G)\to 0} \frac{1}{vol(G)} \iint_{\sigma(G)} \vec{F}\cdot\vec{n}\, dS$$

In an incompressible fluid, points at which div $\vec{F}(P_0) > 0$ are called <u>sources</u> and points at which $\vec{F}(P_0) < 0$ are called <u>sinks</u>. Fluid enters at a source and drains at a sink. In an incompressible fluid w/out sources or sinks we must have <u>div $\vec{F}(P) = 0$</u> at every point P.

<u>Gauss's Law for Electric field</u>: $\Phi = \iint_\sigma \vec{F}\cdot\vec{n}\, dS = 4\pi\left(\frac{Q}{4\pi\epsilon_0}\right) = \frac{Q}{\epsilon_0}$

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4) $\vec{F}(x,y,z) = xy\vec{i} + yz\vec{j} + xz\vec{k}$; σ: cube bounded by x = 0, x = 2, y = 0, y = 2, z = 0, z = 2

$\sigma_1: x=0 \quad \vec{F}\cdot\vec{n} = -xy = 0, \iint_{\sigma_1}(0)\,dA = 0 \qquad \sigma_2: x=2 \quad \vec{F}\cdot\vec{n} = xy = 2y \quad \iint_{\sigma_2} 2y\, dA = \underline{8}$

$\sigma_3: y=0 \quad \vec{F}\cdot\vec{n} = -yz = 0, \iint_{\sigma_3}(0)\,dA = 0 \qquad \sigma_4: y=2 \quad \vec{F}\cdot\vec{n} = yz = 2z \quad \iint_{\sigma_4} 2z\, dA = 8$

$\sigma_5: z=0 \quad \vec{F}\cdot\vec{n} = -xz = 0, \iint_{\sigma_5}(0)\,dA = 0 \qquad \sigma_6: z=2 \quad \vec{F}\cdot\vec{n} = xz = 2x \quad \iint_{\sigma_6} 2x\, dA = 8$

$\iint_\sigma \vec{F}\cdot\vec{n}\, dS = \underline{\underline{24}}$

$$\iiint_G \text{div}\,\vec{F}\, dV = \iiint_G (y+z+x)\, dV = \int_0^2\int_0^2\int_0^2 (y+z+x)\, dz\, dy\, dx = \int_0^2\int_0^2 \left(yz + \tfrac{1}{2}z^2 + xz\right)\Big|_0^2 dy\, dx$$

$$= \int_0^2\int_0^2 (2y + 4 + 2x)\, dy\, dx = \int_0^2 (y^2 + 4y + 2xy)\Big|_0^2 dx = \int_0^2 (4 + 8 + 4x)\, dx$$

$$= (12x + 2x^2)\Big|_0^2 =$$

7) $\vec{F}(x,y,z) = (x-z)\vec{i} + (y-x)\vec{j} + (z-y)\vec{k}$; σ: $x^2+y^2=a^2$, $z=0$, $z=1$

$$\iiint \operatorname{div} F\, dV = \iiint_G (1+1+1)\, dV = 3\iiint_G dV = 3(\text{vol cylinder}) = 3[\pi r^2 h] = 3\pi(a)^2(1)$$

$$= \underline{\underline{3a\pi^2}}$$

12) $\vec{F}(x,y,z) = 2xz\vec{i} + yz\vec{j} + z^2\vec{k}$; σ: $z=\sqrt{a^2-x^2-y^2}$, below by xy-plane

$$\iiint (2z+z+2z)\, dV = \iiint 5z\, dV = 5\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho\cos\phi\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$5\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3\cos\phi\sin\phi\, d\rho\, d\phi\, d\theta$$

$$\frac{5}{4}a^4\int_0^{2\pi}\int_0^{\pi}\cos\phi\sin\phi\, d\phi\, d\theta \qquad u=\cos\phi,\quad du=-\sin\phi\, d\phi$$

$$\frac{5}{8}a^4\int_0^{2\pi} -\cos^2\phi\Big|_0^{\pi}\, d\theta$$

$$\frac{5}{8}a^4\int_0^{2\pi} d\theta = \underline{\underline{\frac{5a^4\pi}{4}}}$$

21) $$\iint_\sigma (f\nabla g)\cdot\vec{n}\, dS = \iiint_G (f\nabla^2 g + \nabla f\cdot\nabla g)\, dV$$

$$\iint_\sigma (f\nabla g)\cdot\vec{n}\, dS = \iiint_G \operatorname{div}(f\nabla g)\, dV = \iiint_G \nabla(f\nabla g)\, dV$$

$$\underline{\underline{\iiint_G [f\nabla^2 g + \nabla f\,\nabla g]\, dV}}$$

28) $\vec{F}(x,y,z) = xy\vec{i} - xy\vec{j} + y^2\vec{k}$

$\operatorname{div}\vec{F} = y - x$

sources: $y > x$

sinks: $\underline{\underline{y < x}}$