Ch. 14 Partial Derivatives

14.1

The independent variables of a fn of 2 or more variables may be restricted to lie in some set D, called the domain.

A fn f of 2 variables, x & y, is a rule that assigns a unique real number $f(x,y)$ to each point (x,y) in some set D in the xy-plane.

A fn of 3 variables, x, y, z, is a rule that assigns a unique real number $f(x,y,z)$ to each point (x,y,z) in some set D in 3-dimensional space.

If f is a fn of two variables, we define the graph of $f(x,y)$ in xyz-space to be the graph of the eqn $z=f(x,y)$. In general, such a graph will be a surface in 3 space.

The projection of the intersection of $z=f(x,y)$ w/ the plane $z=k$ onto the xy-plane is called the level curve of height k.

A set of level curves is called a contour plot.

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6) $g[u(x,y), v(x,y)]$

$g(u,v) = v \sin(u^2 y)$

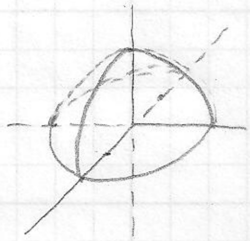
$u(x,y) = x^2 y^3$

$v(x,y) = \pi x y$

$g(x,y) = \pi x y \sin(\pi x^5 y^7)$

26) $f(x,y) = -\sqrt{9-(x^2+y^2)}$

$z = \sqrt{9-(x^2+y^2)}$



14) $f(x,y,z) = zxy + x$

a) $f(x+y,\ x-y,\ x^2)$

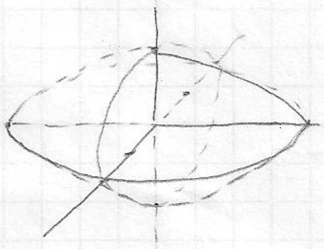
$f(x,y,z) = (x^2)(x+y)(x-y) + (x+y)$

$f(x,y,z) = x^2(x^2-y^2)+(x+y)$

$= x^4 - x^2y^2 + x + y$

45) $f(x,y,z) = 4x^2+y^2+4z^2$; $k=16$

$1 = \dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{4}$



23a) $f(x,y) = x e^{-\sqrt{y+2}}$

$y \geq -2$

all pts on & above

$y=-2$

Ch. 14

14.2

If C is a smooth parametric curve in 2- or 3-space, then the limits of the curve are defined by

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{along } C}} f(x,y) = \lim_{t\to t_0} f(x(t),y(t)) \qquad \lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ \text{along } C}} f(x,y,z) = \lim_{t\to t_0} f(x(t),y(t),z(t))$$

defn 14.2.1

Let f be a fn of 2 variables, and assume that f is defined at all pts w/in a disk centered at (x_0,y_0), except possibly at (x_0,y_0). We will write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x,y)$ satisfies $|f(x,y)-L| < \epsilon$ whenever the distance between (x,y) & (x_0,y_0) satisfies $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

Thm. 14.2.2

(a) If $f(x,y)\to L$ as $(x,y)\to(x_0,y_0)$, then $f(x,y)\to L$ as $(x,y)\to(x_0,y_0)$ along any smooth curve

(b) If the limit of $f(x,y)$ fails to exist as $(x,y)\to(x_0,y_0)$ along some smooth curve or if $f(x,y)$ has different limits as $(x,y)\to(x_0,y_0)$ along two different smooth curves, then the limit of $f(x,y)$ d.n.e. as $(x,y)\to(x_0,y_0)$.

Defn 14.2.3

A fn $f(x,y)$ is said to be continuous @ (x_0,y_0) if $f(x_0,y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Thm 14.2.4

(a) If $g(x)$ is continuous @ x_0 & $h(y)$ is continuous @ y_0, then $f(x,y) = g(x)h(y)$ is continuous at (x_0,y_0)

(b) If $h(x,y)$ is continuous @ (x_0,y_0) and $g(u)$ is continuous at $u = h(x_0,y_0)$, then the composition $f(x,y) = g(h(x,y))$ is continuous at (x_0,y_0)

c) If $f(x,y)$ is continuous at (x_0,y_0) and if $x(t)$ & $y(t)$ are continuous at t_0 w/ $x(t_0)=x_0$ & $y(t_0)=y_0$ then the composition $f(x(t),y(t))$ is continuous at t_0.

A fn of 2-variables that is continuous at every point (x,y) in the xy-plane is said to be continuous everywhere.

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defn 14.2.5

Let R denote a subset of the xy-plane contained w/in the domain of a fn $f(x,y)$. We say the $f(x,y)$ is continuous on R provided that for every point (x_0, y_0) in R & for every $\epsilon > 0$ $\exists$ a $\delta > 0 \ni f(x,y)$ satisfies $|f(x,y) - f(x_0,y_0)| < \epsilon$ whenever (x,y) is in R & the distance between (x,y) & (x_0,y_0) satisfies

$$0 \le \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

Extension to 3-space

defn 14.2.6 (paraphrased)

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L \text{ if given } \epsilon > 0 \ \exists \ \delta > 0 \ni |f(x,y,z) - L| < \epsilon$$

whenever $0 < \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} < \delta$

P.940

6) $\lim_{(x,y)\to(4,-2)} x\sqrt[3]{y^3+2x} = 4\sqrt[3]{-8+8} = \underline{\underline{0}}$

8b) $\lim_{(x,y)\to(0,0)} \frac{\cos xy}{x+y}$

along $x=0$ $\lim_{y\to 0} \frac{1}{y} = \underline{\underline{\text{d.n.e.}}}$

14) $\lim_{(x,y)\to(0,0)} \frac{x^4-16y^4}{x^2+4y^2}$

$\lim_{(x,y)\to(0,0)} \frac{(x^2-4y^2)\cancel{(x^2+4y^2)}}{\cancel{x^2+4y^2}} = \underline{\underline{0}}$

18) $\lim_{(x,y,z)\to(2,0,-1)} \ln(2x+y-z) = \underline{\underline{\ln 5}}$

22) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sqrt{x^2+y^2}}$

$\lim_{r\to 0^+} \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{r}$

$\lim_{r\to 0^+} r^3\cos^2\theta\sin^2\theta = \underline{\underline{0}}$

32) $f(x,y) = xy\ln(x^2+y^2)$. Can $f(x,y)$ be redef so f will be continuous @ $(0,0)$?

$f(r,\theta) = |r^2\cos\theta\sin\theta\ln r^2|$, $r > 0$ because $r = \sqrt{x^2+y^2}$

$|r^2\cos\theta\sin\theta\, 2\ln r| \le |2r^2\ln r|$

but $\lim_{r\to 0^+} |2r^2\ln r| = 0$ because

$\lim_{r\to 0^+} 2r^2\ln r = 0\cdot\infty$

$\lim_{r\to 0^+} \frac{\ln r}{\frac{1}{2r^2}} = \frac{\infty}{\infty}$

$\lim_{r\to 0^+} \frac{\frac{1}{r}}{-\frac{1}{4r^3}} = 0$

$\therefore f(x,y)$ will be continuous by redefining it as

$$f(x,y) = \begin{cases} xy\ln(x^2+y^2) & (x,y) \ne (0,0) \\ 0 & (x,y) = 0 \end{cases}$$

44) $f(x,y,z) = \sin\sqrt{x^2+y^2+3z^2}$

<u>continuous</u> on all of 3-space

14.3 Partial Derivatives

$f_x(x_0, y_0) = \frac{d}{dx} f(x, y_0)\Big|_{x=x_0}$ $\qquad$ $f_y(x_0, y_0) = \frac{d}{dy} f(x_0, y)\Big|_{y=y_0}$

$f_x(x_0, y_0)$ is the slope of the surface in the x-direction; $f_y(x_0, y_0)$ is the slope of the surface in the y-direction at (x_0, y_0)

For a fn of three variables, there are three partial derivatives:

$f_x(x, y, z)$, $f_y(x, y, z)$, & $f_z(x, y, z)$

Higher order partial derivatives

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = f_{xx}$ $\qquad$ $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = f_{yy}$

* $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right) = f_{xy}$ $\qquad$ * $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = f_{yx}$

Note: The mixed second order partials have differing order of operations depending on the notation used

when using $\frac{\partial f}{\partial \square \partial \bigcirc}$ read the order from right to left in the denominator

when using $f_{\square\bigcirc}$ subscript notation, read the order left to right

From Thm 14.3.2, paraphrased, it says the mixed second order partial derivatives are equal. $\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}$ or $f_{xy} = f_{yx}$. This is (usually) easily verifiable.

p. 949, eqn 6 $\quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial t^2}$ This is called the wave eqn. It is classified as a partial differential eqn. Techniques for solving these aren't taught until a course called Partial Differential Equations (PDE). It's a really cool course. Take it as an elective if not a required one for your major.

p.2 14.3

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12) $z = \cos(x^5y^4)$

$\frac{\partial z}{\partial x} = -5x^4y^4\sin(x^5y^4)$ $\frac{\partial z}{\partial y} = -4x^5y^3\sin(x^5y^4)$

18) $f(x,y) = \frac{x+y}{x-y}$

$f_x(x,y) = \frac{(x-y)-(x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$

$f_y(x,y) = \frac{(x-y)+(x+y)}{(x-y)^2} = \frac{2x}{(x-y)^2}$

28) $w = x^2y\cos z$

a) $\frac{\partial w}{\partial x} = 2xy\cos z$ b) $\frac{\partial w}{\partial y} = x^2\cos z$ c) $\frac{\partial w}{\partial z} = -x^2y\sin z$

d) $\frac{\partial w}{\partial x}\Big|_{(2,x,y)} = 4y\cos z$ e) $\frac{\partial w}{\partial y}\Big|_{(2,1,z)} = 4\cos z$ f) $\frac{\partial w}{\partial z}\Big|_{(2,1,0)} = 0$

46) $V = \frac{\pi}{24}d^2\sqrt{4s^2-d^2}$ s = slant height, d = diameter

a) $\frac{\partial V}{\partial s} = \frac{\pi}{\cancel{24}_6}d^2\left[\frac{1}{\cancel{2}}(4s^2-d^2)^{-\frac{1}{2}}(\cancel{8}^4 s)\right] = \frac{\pi}{6}d^2s(4s^2-d^2)^{-\frac{1}{2}}$

b) $\frac{\partial V}{\partial d} = \frac{\pi}{24}d^2\left[\frac{1}{\cancel{2}}(4s^2-d^2)^{-\frac{1}{2}}(-\cancel{2}d)\right] + \frac{2\pi}{24}d(4s^2-d^2)^{\frac{1}{2}} = \frac{d\pi}{24}(4s^2-d^2)^{-\frac{1}{2}}[8s^2-3d^2]$

c) $\frac{\partial V}{\partial s}\Big|_{(s=10,\ d=16)} = \frac{2560\pi}{6}\left(\frac{1}{12}\right) = \frac{320}{9}\pi$

d) $\frac{\partial V}{\partial d}\Big|_{(s=10,d=16)} = \frac{\cancel{16}^4\pi}{\cancel{24}_3}\left(\frac{1}{\cancel{12}_3}\right)(\cancel{32}^4) = \frac{16\pi}{9}$

52) $(1,3,3)$, $z = x^2y$ intersection w/

a) the plane $x=1$

$\frac{\partial z}{\partial y} = x^2$ $\frac{\partial z}{\partial y}\Big|_{(1,3)} = 1 \therefore \vec{j}+\vec{k}$ is parallel to the tangent line so eqns are

$x = 1+0t,\ y = 3+1t,\ z = 3+1t$

b) the plane $y=3$

$\frac{\partial z}{\partial x} = 2xy$ $\frac{\partial z}{\partial x}\Big|_{(1,3)} = 6 \therefore \vec{i}+6\vec{k}$ is parallel to the tangent line so eqns are

$x = 1+1t,\ y = 3+0t,\ z = 3+6t$

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60) $\ln(2x^2+y-z^3+3w)=7$

$$\frac{\partial w}{\partial x}: \frac{1}{2x^2+y-z^3+3w}\left(4x+3\frac{\partial w}{\partial x}\right)=0 \Rightarrow \frac{\partial w}{\partial x} = -\frac{4x}{3}$$

$$\frac{\partial w}{\partial y}: \frac{1}{2x+y-z^3+3w}\left(1+3\frac{\partial w}{\partial y}\right)=0 \Rightarrow \frac{\partial w}{\partial y} = -\frac{1}{3}$$

$$\frac{\partial w}{\partial z}: \frac{1}{2x+y-z^3+3w}\left(-3z^2+3\frac{\partial w}{\partial z}\right)=1 \Rightarrow \frac{\partial w}{\partial z} = \frac{2x+y-z^3+3z^2+3w}{3}$$

85a) $z = x^2-y^2+2xy$

$\frac{\partial z}{\partial x} = 2x+2y$ $\qquad$ $\frac{\partial z}{\partial y} = -2y+2x$

$\frac{\partial^2 z}{\partial x^2} = 2$ $\qquad$ $\frac{\partial^2 z}{\partial y^2} = -2$ $\qquad$ $\Rightarrow 2-2=0$

Ch. 14

14.4

If a fn $f(x,y)$ of 2 variables is differentiable at a point (x_0,y_0), we want it to be the case that

- the surface $z=f(x,y)$ has a nonvertical tangent plane at the point $(x_0,y_0,f(x_0,y_0))$

- the values of f at points near (x_0,y_0) can be very closely approximated by the values of a linear fn

- f is continuous at (x_0,y_0)

14.4.1

A fn f of one variable is said to be <u>differentiable</u> at x_0 provided $\exists$ a linear approximation $L(x)=f(x_0)+m(x-x_0)$ to f at x_0 for which the error $E(x)=f(x)-L(x)$ satisfies

$$\lim_{x\to x_0}\frac{E(x)}{|x-x_0|}=0$$

When f is differentiable at x_0, we denote the number m by $f'(x_0)$ and refer to it as the derivative of f at x_0

Any linear fn $L(x,y)$ whose graph is a plane through P can be written in the form
$L(x,y)=f(x_0,y_0)+m_1(x-x_0)+m_2(y-y_0)$

We refer to a fn $L(x,y)$ as a <u>linear approximation</u> to f @ (x_0,y_0).

$E(x,y)=f(x,y)-L(x,y)=f(x,y)-[f(x_0,y_0)+m_1(x-x_0)+m_2(y-y_0)]$ is the error that results if $L(x,y)$ is used to approximate $f(x,y)$

14.4.2

A fn of 2 variables is said to be differentiable at (x_0,y_0) provided $f_x(x_0,y_0)$ & $f_y(x_0,y_0)$ both exist and

$$\lim_{(x,y)\to(x_0,y_0)}\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$$

where $E(x,y)=f(x,y)-L(x,y)$ denotes error in the linear approximation $L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$ to f @ (x_0,y_0). This L is the local linear approximation to f @ (x_0,y_0)

14.4.3

The defn extended to $f(x,y,z)$

14.4.4

If a fn is differentiable at a point, then it's continuous at that point

14.4.5

If all 1st order partial derivatives of f exist & are continuous at a point, then f is differentiable at that point

p.2 14.4

If $z = f(x,y)$ is differentiable at a point (x_0, y_0), we let
$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$ denote a new fn w/ dependent variable z & independent

variables x & y. This is referred to as the total differential of z @ (x_0, y_0) or as the total differential of f @ (x_0, y_0)

Similarly for a fn $w = f(x,y,z)$ of 3 variables we have the total differential of w @ (x_0, y_0, z_0)
$$dw = f_x(x,y,z)\,dx + f_y(x,y,z)\,dy + f_z(x,y,z)\,dw$$

In the 2-variable case, the approximation $f(x,y) \approx L(x,y)$ can be written as
$$\Delta f \approx f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y$$
$$\Delta z \approx \Delta f$$

P. 961

8) $P(0,-1)$ $f_y(0,-1) = -2$; $f(0,-1) = 3$; if $L(0.1,-1.1) = 3.3$, find $f_x(0,-1)$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$
$$3.3 = 3 + f_x(0,-1)(.1) - 2(-.1)$$
$$.3 = (.1)\,f_x(0,-1) + .2$$
$$.1 = .1\,f_x(0,-1)$$
$$\underline{1 = f_x(0,-1)}$$

16) $f(x,y,z) = xyz$; $L(x,y,z) = x - y - z - 2$

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

$$\cancel{x} - y - \cancel{z} - 2 = x_0y_0z_0 + y_0z_0(x-x_0) + x_0z_0(y-y_0) + x_0y_0(z-z_0)$$

$\therefore\ 1 = y_0z_0 \quad -1 = x_0z_0 \quad -1 = x_0y_0 \quad -2 = x_0y_0z_0 - 3x_0y_0z_0$
$1 = x_0y_0z_0$

$\Rightarrow x_0 = -y_0 = -z_0$
$\Rightarrow x_0 = 1,\ y_0 = -1,\ z_0 = -1$
$\underline{P(1,-1,-1)}$

20) $f(x,y) = \ln xy$; $P(1,2)$; $Q(1.01, 2.02)$

a) $f(P) = \ln 2$; $f_x(P) = 1$ $f_y(P) = 1/2$
$\underline{L(x,y) = \ln 2 + 1(x-1) + \tfrac{1}{2}(y-2)}$

b) $L(Q) - f(Q) = \ln 2 + 1(.01) + \tfrac{1}{2}(.02) - \ln(1.01)(2.02) \approx \underline{.000099338}$

$|PQ| = \sqrt{(1.01-1)^2 + (2.02-2)^2} \approx \underline{.02236}$

$$\frac{|L(Q) - f(Q)|}{|PQ|} \approx \underline{.00444}$$

p.3 14.4

p. 962

26) $z = e^{xy}$

$dz = ye^{xy}\,dx + xe^{xy}\,dy$

34) $w = 4x^2y^3z^2 - 3xy + z + r$

$dw = (8xy^3z^2 - 3y)\,dx + (12x^2y^2z^2 - 3x)\,dy + (8x^2y^3z + 1)\,dz$

38) $f(x,y) = x^{1/3}y^{1/2}$; $P(8,9)$, $Q(7.78, 9.03)$

$df = \frac{1}{3}x^{-2/3}y^{1/2}\,dx + \frac{1}{2}x^{1/3}y^{-1/2}\,dy$; $dx = -.22$, $dy = .03$

$df = -.045$ Δf

50) $P = \frac{kT}{V}$ $\frac{dT}{T} = .03$, $\frac{dV}{V} = .05$ $\frac{dP}{P}$

$dP = \frac{k}{V}\,dT - \frac{kT}{V^2}\,dV$

$\frac{dP}{P} = \frac{\frac{k}{V}\left(dT - \frac{T}{V}\,dV\right)}{\frac{kT}{V}} = \frac{dT}{T} - \frac{dV}{V} = .03 - .05 = -.02 \Rightarrow$ decrease by 2%

Ch. 4

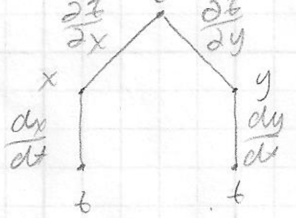
14.5 Chain Rule

2 Variable chain rule - If $x = x(t)$ & $y = y(t)$ are differentiable @ t, and if $z = f(x,y)$ is differentiable at the point $(x,y) = [x(t), y(t)]$, then $z = f[x(t), y(t)]$ is differentiable @ t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad *$$

where the ordinary derivatives are evaluated at t and the partials at (x,y).

* This can be represented by a diagram



There are many variations in derivative notation

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \qquad \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad \frac{df}{dt} = f_x\, x'(t) + f_y\, y'(t)$$

Three variable chain rule - extending the two variable rule to include a third variable $z(t)$, it becomes

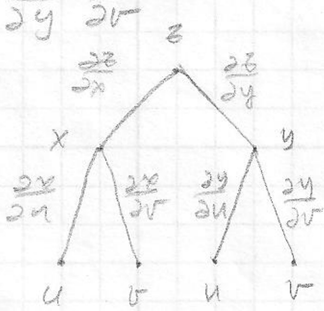
$$w = f(x,y,z) \qquad \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

If the eqn $f(x,y)=c$ defines y implicitly as a differentiable fn of x, and if $\partial f/\partial y \neq 0$, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

Two Variable Chain - This time $x = x(u,v)$, $y = y(u,v)$, $z = f(x,y)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad \& \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$



Three variable version w/ $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ & $w = f(x,y,z)$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \quad \& \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

p. 970

2) $z = \ln(2x^2+y)$; $x = t^{1/2}$, $y = t^{2/3}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{4x}{2x^2+y}\cdot\frac{1}{2}t^{-1/2} + \frac{1}{2x^2+y}\cdot\frac{2}{3}t^{-1/3}$$

$$\frac{dz}{dt} = \frac{{}^{2}\cancel{4t^{1/2}}}{2t+t^{2/3}}\cdot\frac{1}{\cancel{2t^{1/2}}} + \frac{1}{2t+t^{2/3}}\cdot\frac{2}{3t^{1/3}}$$

$$\frac{dz}{dt} = \frac{2}{2t+t^{2/3}} + \frac{2}{3t^{1/3}(2t+t^{2/3})}$$

$$\frac{dz}{dt} = \frac{6t^{1/3}+2}{3t^{1/3}(2t+t^{2/3})}$$

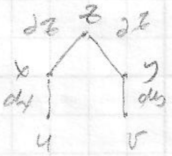
8) $w = \ln(3x^2-2y+4z^3)$; $x = t^{1/2}$, $y = t^{2/3}$, $z = t^{-2}$

$$\frac{dw}{dt} = \frac{6x}{3x^2-2y+4z^3}\cdot\frac{1}{2}t^{-1/2} + \frac{-2}{3x^2-2y+4z^3}\cdot\frac{2}{3}t^{-1/3} + \frac{12z^2}{3x^2-2y+4z^3}\cdot -2t^{-3}$$

$$\frac{dw}{dt} = \frac{{}^{3}\cancel{6t^{1/2}}}{3t-2t^{2/3}+4t^{-6}}\cdot\frac{1}{\cancel{2t^{1/2}}} - \frac{4}{3t^{1/3}(3t-2t^{2/3}+4t^{-6})} - \frac{24t^{-4-7}}{t^{3}(3t-2t^{2/3}+4t^{-6})}$$

$$\frac{dw}{dt} = \frac{9t^{1/3} - 4 - 72t^{-20/3}}{3t^{1/3}(3t-2t^{2/3}+4t^{-6})}$$

15) $z = \frac{x}{y}$, $x = 2\cos u$, $y = 3\sin v$



$$\frac{\partial z}{\partial u} = \frac{1}{y}\cdot -2\sin u + -\frac{x}{y^2}\cdot 0$$

$$\frac{\partial z}{\partial u} = -\frac{2\sin u}{3\sin v}$$

$$\frac{\partial z}{\partial v} = \frac{1}{y}\cdot 0 + -\frac{x}{y^2}\cdot 3\cos v = -\frac{3\cos u\cos v}{9\sin^2 v} = -\frac{\cos u\cos v}{3\sin^2 v}$$

26) $w = 3xy^2z^3$, $y = 3x^2+2$, $z = \sqrt{x-1}$

$$\frac{dw}{dx} = 3y^2z^3 + 6xyz^3(6x) + 9xy^2z^2\left(\frac{1}{2}(x-1)^{-1/2}\right)$$

$$\frac{dw}{dx} = 3(3x^2+2)(x-1)^{3/2} + 36x^2(3x^2+2)(x-1)^{3/2} + \frac{9x(3x^2+2)(x-1)^{1/2}}{2\cancel{(x-1)^{1/2}}}$$

$$\frac{dw}{dx} = 3(3x^2+2)(x-1)^{1/2}\left[x-1+12x^2(x-1)+\frac{9x}{2}\right] = 3(3x^2+2)(x-1)^{1/2}\left[\frac{2x-2+24x^3-24x^2+9x}{2}\right]$$

$$\frac{dw}{dx} = 3(3x^2+2)(x-1)^{1/2}\left[\frac{24x^3-24x^2+11x-2}{2}\right]$$

32) $\frac{dy}{dx} = \frac{\partial f/\partial x}{\partial f/\partial y}$; $x^3 - 3xy^2 + y^3 = 5$

$$\frac{dy}{dx} = \frac{3x^2-3y^2}{-6xy+3y^2} = -\frac{x^2-y^2}{2xy-y^2}$$

p.3 14.5

46) velocity $\vec{v} = \hat{i} - 4\hat{j}$ cm/s @ $(3,2)$ $T(x,y) = y^2 \ln x$, $x \geq 1$ find dT/dt @ $(3,2)$

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial T}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{dT}{dt} = \frac{y^2}{x}\cdot\frac{dx}{dt} + 2y\ln x\cdot\frac{dy}{dt}$$ remember, $\frac{dx}{dt} = 1$ & $\frac{dy}{dt} = -4$ @ $(3,2)$ from $\vec{v}$

$$\frac{dT}{dt} = \frac{4}{3}(1) + 2(2)\ln 3(-4)$$

$$\frac{dT}{dt} = \left(\frac{4}{3} - 16\ln 3\right)^{\circ} C/sec \approx -16.244^{\circ} C/sec$$

57) Let $z = f(x-y,\, y-x)$. Show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

let $u = x-y$, $v = y-x$ $\therefore z = f(u,v)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \quad \& \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot(1) + \frac{\partial z}{\partial v}\cdot(-1) \quad\quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot(-1) + \frac{\partial z}{\partial v}\cdot(1)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \quad\quad \frac{\partial z}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\therefore \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = 0$$

Ch. 14

14.6

If $f(x,y)$ is a fn of x & y, and if $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is a unit vector, then the directional derivative of f in the direction of $\vec{u}$ @ (x_0, y_0) is denoted by $D_{\vec{u}} f(x_0, y_0)$ and is defined by

$$D_{\vec{u}} f(x_0, y_0) = \frac{d}{ds}\Big[f(x_0 + su_1, y_0 + su_2)\Big]_{s=0}$$

Geometrically, $D_{\vec{u}} f(x_0, y_0)$ is the slope of the surface $z = f(x,y)$ in the direction of $\vec{u}$ @ $(x_0, y_0, f(x_0, y_0))$

Analytically, $D_{\vec{u}} f(x_0, y_0)$ represents the instantaneous rate of change of $f(x,y)$ wrt distance in the direction of $\vec{u}$ @ (x_0, y_0)

For 3-space w/ $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ & $f(x,y,z)$ a fn of $x, y,$ & z, the directional derivative becomes

$$D_{\vec{u}} f(x_0, y_0, z_0) = \frac{d}{ds}\Big[f(x_0 + su_1, y_0 + su_2, z_0 + su_3)\Big]_{s=0}$$

If $f(x,y)$ is differentiable at (x_0, y_0), and if $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is a unit vector, then the directional derivative $D_{\vec{u}} f(x_0, y_0)$ exists and is given by

$$D_{\vec{u}} f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\, u_1 + \frac{\partial f}{\partial y}(x_0, y_0)\, u_2 \qquad (4)$$

If $f(x,y,z)$ is diff. @ (x_0, y_0, z_0) & $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ is a unit vector, then

$$D_{\vec{u}} f(x_0, y_0, z_0) = \frac{\partial}{\partial x} f(x_0, y_0, z_0)\, u_1 + \frac{\partial}{\partial y} f(x_0, y_0, z_0)\, u_2 + \frac{\partial}{\partial z} f(x_0, y_0, z_0)\, u_3 \qquad (5)$$

Recall from 12.2 that a unit vector $\vec{u}$ in the xy-plane can be expressed as $\vec{u} = \cos\phi\,\vec{i} + \sin\phi\,\vec{j}$ where ϕ is the angle from the positive x-axis to $\vec{u}$

$\therefore$ (4) can be expressed as $D_{\vec{u}} f(x_0, y_0) = \frac{\partial}{\partial x} f(x_0, y_0)\cos\phi + \frac{\partial}{\partial y} f(x_0, y_0)\sin\phi$

If f is a fn of x & y, then the gradient of f is defined by

$$\nabla f(x,y) = \frac{\partial}{\partial x} f(x,y)\,\vec{i} + \frac{\partial}{\partial y} f(x,y)\,\vec{j}$$

If f is a fn of $x, y,$ & z, then the gradient of f is defined by

$$\nabla f(x,y,z) = \frac{\partial}{\partial x} f(x,y,z)\,\vec{i} + \frac{\partial}{\partial y} f(x,y,z)\,\vec{j} + \frac{\partial}{\partial z} f(x,y,z)\,\vec{k}$$

(4) & (5) Can now be written, respectively, as

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

$$D_{\vec{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \vec{u}$$

Properties of ∇f

if $\nabla f \neq \vec{0}$,

$\nabla f(x_0, y_0) \cdot \vec{u} = \| \nabla f(x, y) \| \cos\theta$; a max occurs when $\theta = 0$

Geometrically, the surface $z = f(x,y)$ has its max slope at a pt (x,y) in the direction of the gradient, and the max slope is $\| \nabla f(x,y) \|$

a min occurs when $\theta = \pi$

Geometrically, the surface $z = f(x,y)$ has its min slope at a point (x,y) in the direction that is opposite to the gradient, and the min slope is $-\| \nabla f(x,y) \|$

If $\nabla f(x,y) = \vec{0}$, $D_{\vec{u}} f(x,y) = 0$ in all directions @ (x,y). This typically occurs when the surface $z = f(x,y)$ has a relative max, a relative min, or a saddle point

Let f be a fn of either 2 or 3 variables, and let P denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$, respectively. assume the f is differentiable at P.

a) If $\nabla f = \vec{0}$ @ P, then all $D_{\vec{u}} f$ @ $P = 0$

b) If $\nabla f \neq \vec{0}$ @ P, then among all possible $D_{\vec{u}} f$ @ P, the derivative in the direction of ∇f @ P has the largest value. The value is $\| \nabla f \|$ @ P.

c) If $\nabla f \neq \vec{0}$ @ P, then among all possible $D_{\vec{u}} f$ @ P, the derivative in the direction opposite of ∇f @ P has the smallest value. The value is $-\| \nabla f \|$ @ P.

Assume that $f(x,y)$ has continuous 1st order partials in a open disk centered @ (x_0, y_0) and that $\nabla f(x_0, y_0) \neq \vec{0}$. Then $\nabla f(x_0, y_0)$ is normal to the level curve of f through (x_0, y_0)

p.3 14.6

P.982

6) $f(x,y,z) = ye^{xz} + z^2$; $P(0,2,3)$; $\vec{u} = \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$

$D_{\vec{u}}f = yze^{xz}\left(\frac{2}{7}\right) + e^{xz}\left(-\frac{3}{7}\right) + \left(xye^{xz} + 2z\right)\left(\frac{6}{7}\right)$

$D_{\vec{u}}f(0,2,3) = \frac{12}{7} + \left(-\frac{3}{7}\right) + \frac{36}{7} = \underline{\underline{\frac{45}{7}}}$

12) $f(x,y) = e^x \cos y$; $P(0, \pi/4)$; $\vec{a} = 5\hat{i} - 2\hat{j} \Rightarrow \vec{u} = \frac{5}{\sqrt{29}}\hat{i} - \frac{2}{\sqrt{29}}\hat{j}$

$D_{\vec{u}}f = e^x \cos y\left(\frac{5}{\sqrt{29}}\right) - e^x \sin y\left(-\frac{2}{\sqrt{29}}\right)$

$D_{\vec{u}}f(0, \pi/4) = \frac{5\sqrt{2}}{2\sqrt{29}} + \frac{2\sqrt{2}}{2\sqrt{29}} = \underline{\underline{\frac{7\sqrt{2}}{2\sqrt{29}}}} = \underline{\frac{7}{\sqrt{58}}}$

20) $f(x,y) = \frac{x-y}{x+y}$; $P(-1,-2)$; $\theta = \frac{\pi}{2}$

$\nabla f(x,y) = \left[\frac{(x+y)-(x-y)}{(x+y)^2}\right]\hat{i} + \left[\frac{-(x+y)-(x-y)}{(x+y)^2}\right]\hat{j} \Rightarrow \frac{2y}{(x+y)^2}\hat{i} - \frac{2x}{(x+y)^2}\hat{j}$

$\nabla f(-1,-2) = -\frac{4}{9}\hat{i} + \frac{2}{9}\hat{j}$; $\vec{u} = 0\hat{i} + 1\hat{j}$

$\nabla f(-1,-2) \cdot \vec{u} = \underline{\underline{\frac{2}{9}}}$

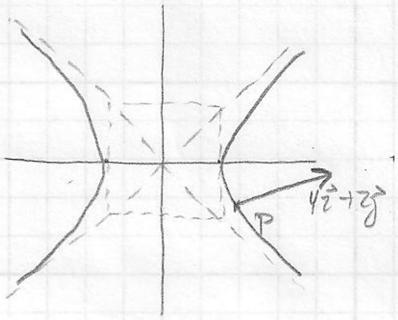
34) $z = e^{-3y}\cos 4x$

$\underline{\nabla z = -4e^{-3y}\sin 4x\,\vec{i} - 3e^{-3y}\cos 4x\,\vec{j}}$

44) $f(x,y) = x^2 - y^2$; $P(2,-1)$; $f(2,-1) = 3$

$\nabla f = 2x\vec{i} - 2y\vec{j}$

$\nabla f(2,-1) = 4\vec{i} + 2\vec{j}$



p. 4 14.6

60) $f(x,y,z) = 4e^{xy}\cos z$; $P(0,1,\pi/4)$

$\nabla f = 4ye^{xy}\cos z\,\vec{i} + 4xe^{xy}\cos z\,\vec{j} - 4e^{xy}\sin z\,\vec{k}$

$\nabla f(0,1,\pi/4) = \sqrt{2}\,\vec{i} + 0\,\vec{j} - \sqrt{2}\,\vec{k}$

$\vec{u}(0,1,\pi/4) = \frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{k}$

to decrease $\underline{\vec{u} = -\frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{k}}$

$-\|\nabla f(0,1,\pi/4)\| = \underline{-4}$

72) $T(x,y,z) = \dfrac{xyz}{1+x^2+y^2+z^2}$

a) $\nabla T = \dfrac{yz(1+x^2+y^2+z^2) - 2x^2yz}{(1+x^2+y^2+z^2)^2}\vec{i} + \dfrac{xz(1+x^2+y^2+z^2) - 2xy^2z}{(1+x^2+y^2+z^2)^2}\vec{j} + \dfrac{xy(1+x^2+y^2+z^2) - 2xyz^2}{(1+x^2+y^2+z^2)^2}\vec{k}$

$P(1,1,1)$

$\nabla T(1,1,1) = \frac{2}{16}\vec{i} + \frac{2}{16}\vec{j} + \frac{2}{16}\vec{k} = \frac{1}{8}\vec{i} + \frac{1}{8}\vec{j} + \frac{1}{8}\vec{k}$; $\vec{a} = -1\vec{i} - 1\vec{j} - 1\vec{k}$

$\vec{u} = -\frac{1}{\sqrt{3}}\vec{i} - \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}$ $\quad D_{\vec{u}}T = \nabla T \cdot \vec{u} = -\frac{3}{8\sqrt{3}} = \underline{-\frac{\sqrt{3}}{8}}$

b) $\vec{u} = \dfrac{\nabla T(1,1,1)}{\|\nabla T(1,1,1)\|} = \dfrac{\frac{1}{8}(\vec{i}+\vec{j}+\vec{k})}{\sqrt{3}/8} = \underline{\frac{1}{\sqrt{3}}(\vec{i}+\vec{j}+\vec{k})}$

c) $D_u T = \left(\frac{1}{8}\vec{i} + \frac{1}{8}\vec{j} + \frac{1}{8}\vec{k}\right)\cdot\left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}\right) = \frac{3}{8\sqrt{3}} = \underline{\frac{\sqrt{3}}{8}}$

Assume that the fn $f(x,y)$ is differentiable at (x_0, y_0) and let $P_0(x_0, y_0, f(x_0,y_0))$ denote the corresponding point on the graph of f. Let T denote the graph of the local linear approximation

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

to f @ (x_0,y_0). Then a line is tangent at P_0 to a curve C on the surface $z = f(x,y)$ iff the line is contained in T.

If $f(x,y)$ is differentiable at a point (x_0,y_0), then the tangent plane to the surface $z=f(x,y)$ at the point $P_0(x_0,y_0,f(x_0,y_0))$ [or (x_0,y_0)] is the plane

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

The line through the point P_0 parallel to the vector $\vec{n}$ is perpendicular to the tangent plane. We refer to this line as the <u>normal line</u> to the surface $z=f(x,y)$ at P_0. It can be expressed parametrically as

$$x = x_0 + f_x(x_0,y_0)t, \quad y = y_0 + f_y(x_0,y_0), \quad z = f(x_0,y_0) - t$$

Assume that $F(x,y,z)$ has continuous first-order partial derivatives and let $c = F(x_0,y_0,z_0)$. If $\nabla F(x_0,y_0,z_0) \neq \vec{0}$, then $\nabla F(x_0,y_0,z_0)$ is a normal vector to the surface $F(x,y,z) = c$ at the point $P_0(x_0,y_0,z_0)$, and the tangent plane to this surface at P_0 is the plane w/ eqn

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

<u>P.990</u>

6) $z = x^{1/2} + y^{1/2}$; $P(4,9,5)$

$f(4,9,5) = 5$

$f_x = \frac{1}{2}x^{-1/2} \Rightarrow f_x(4,9,5) = \frac{1}{4}$

$f_y = \frac{1}{2}y^{-1/2} \Rightarrow f_y(4,9,5) = \frac{1}{6}$

$\therefore$

$z = 5 + \frac{1}{4}(x-4) + \frac{1}{6}(y-9)$

$12z = 60 + 3(x-4) + 2(y-9)$

$12z = 60 + 3x - 12 + 2y - 18$

$3x + 2y - 12z = 30$

$x = 4 + \frac{1}{4}t,\ y = 9 + \frac{1}{6}t,\ z = 5 - t$

12) $z = (x^2+y^2)^{1/2}$, $z = \frac{1}{10}(x^2+y^2) + \frac{5}{2}$

@ $(3,4,5)$

① $5 = (3^2+4^2)^{1/2}$ $5 = \frac{1}{10}(3^2+4^2) + \frac{5}{2}$

$5 = 5$ ✓ $5 = \frac{5}{2} + \frac{5}{2}$

$5 = 5$ ✓

$\therefore$ intersect

② $f(x_0,y_0) = 5$ $f(x_0,y_0) = 5$

$f_x = \frac{1}{2}(x^2+y^2)^{-1/2}(2x)$ $f_x = \frac{1}{5}x$

$\Rightarrow \frac{3}{5}$ $\Rightarrow \frac{3}{5}$

$f_y = \frac{1}{2}(x^2+y^2)^{-1/2}(2y)$ $f_y = \frac{1}{5}y$

$\Rightarrow \frac{4}{5}$ $\Rightarrow \frac{4}{5}$

$\therefore \vec{n} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j} - 1\vec{k}$ $\therefore \vec{n} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j} - k$

Since both have the same normal @ $(3,4,5)$ they have a common tangent plane @ $(3,4,5)$.

p.2 14.7

16) $xz - yz^3 + yz^2 = 2$ @ $(2,-1,1)$

a) $\nabla F = z\vec{i} + (-z^3+z^2)\vec{j} + (-3yz^2+2yz)\vec{k}$

$\nabla F(2,-1,1) = 1\vec{i} + 0\vec{j} + 1\vec{k}$

$1(x-2) + 0(y+1) + 1(z-1) = 0$

$\underline{x + z = 3}$

b) $x = \underline{2+1t,\ y = -1+0t},\ z = 1-t$

c) $\cos\theta = \frac{|\nabla F \cdot \vec{k}|}{\|\nabla F\|\|\vec{k}\|} = \frac{1}{\sqrt{2}\cdot 1} = \frac{1}{\sqrt{2}} = \underline{45^\circ}$

23) $z = x^2 + y^2$, $x^2 + 4y^2 + z^2 = 9$ @ $(1,-1,2)$

$f_x = 2x \Rightarrow 2$ $\qquad f_x = 2x \Rightarrow 2$

$f_y = 2y \Rightarrow -2$ $\qquad f_y = 8y \Rightarrow -8$

$\therefore \vec{n}_1 = 2\vec{i} - 2\vec{j} - \vec{k}$ $\qquad f_z = 2z \Rightarrow 4$

$\qquad \therefore \vec{n}_2 = 2\vec{i} - 8\vec{j} + 4\vec{k}$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -1 \\ 2 & -8 & 4 \end{vmatrix} = -16\vec{i} - 10\vec{j} - 12\vec{k} \Rightarrow 8\vec{i} + 5\vec{j} + 6\vec{k}$$

$\therefore x = \underline{1+8t,\ y = -1+5t},\ z = 2+6t$

29) $z = f(x,y)$ & $z = g(x,y)$ intersect @ $P(x_0, y_0, z_0)$; normal line @ P are $\perp$ iff

$f_x(x_0,y_0)g_x(x_0,y_0) + f_y(x_0,y_0)g_y(x_0,y_0) = -1$

$\vec{n}_1 = f_x(x_0,y_0)\vec{i} + f_y(x_0,y_0)\vec{j} - \vec{k}$ & $\vec{n}_2 = g_x(x_0,y_0)\vec{i} + g_y(x_0,y_0)\vec{j} - \vec{k}$

$\vec{n}_1$ & $\vec{n}_2$ $\perp$ iff $\vec{n}_1 \cdot \vec{n}_2 = 0$

$\therefore \vec{n}_1 \cdot \vec{n}_2 = f_x(x_0,y_0)g_x(x_0,y_0) + f_y(x_0,y_0)g_y(x_0,y_0) + 1 = 0$

$\underline{f_x(x_0,y_0)g_x(x_0,y_0) + f_y(x_0,y_0)g_y(x_0,y_0) = -1}$

Relative Max (in two variables) - @ a pt (x_0, y_0) if there is a disk centered at (x_0, y_0) $\ni$ $f(x_0, y_0) \geq f(x, y)$ $\forall$ point (x, y) that lie inside the disk, and f has an absolute max at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ $\forall$ point (x, y) in the domain of f.

Relative Min (in two variables) - @ a pt (x_0, y_0) if there is a disk centered at (x_0, y_0) $\ni$ $f(x_0, y_0) \leq f(x, y)$ $\forall$ point (x, y) that lie inside the disk, and f has an absolute min at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ $\forall$ points (x, y) in the domain of f.

If f has a relative max or relative min, we say f has a relative extremum @ (x_0, y_0). If f has an absolute max or absolute min, then we say f has an absolute extremum at (x_0, y_0).

A point (x_0, y_0) is an interior point of a set D in 2-sp if there is some circular disk w/ positive radius, centered @ (x_0, y_0) and contains only point in D.

A pt (x_0, y_0) is called a boundary point of D if $\forall$ circular disk w/ positive radius and centered @ (x_0, y_0) contain both points inside D and outside D.

In 3-space it becomes a spherical ball instead of disk

In 2- or 3-space a set is called open if it contain none of its boundary points and closed if it contain all of its boundary points

A set in 2-space is bounded if the entire set can be contained in some rectangle and unbounded if there is no rectangle that can contain all points

In 3-space a set is bounded if the entire set can be contained in some box and unbounded if it cannot.

Extreme Value Theorem - If $f(x, y)$ is continuous on a closed & bounded set R, then it has both an absolute max & absolute min.

If f has relative extremum at a point (x_0, y_0), and if the first order partials of f exist at (x_0, y_0), then

$$f_x(x_0, y_0) = 0 \quad \& \quad f_y(x_0, y_0) = 0$$

p.2 14.8

A point (x_0, y_0) in the domain of a fn $f(x,y)$ is called a critical point of the fn if $f_x(x_0,y_0)=0$ & $f_y(x_0,y_0)=0$ or if one or both partials do not exist.

A surface $z=f(x,y)$ has a saddle point @ (x_0,y_0) if there are two distinct vertical plane through this point ∋ the trace of the surface in one plane has a relative max @ (x_0,y_0) and the trace in the other has a relative minimum @ (x_0,y_0)

Second Partials Test - Let f be a fn of 2 variables w/ continuous 2nd order partials in some disk centered at a critical point (x_0,y_0) and let

$$D = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0)$$

a) If $D>0$ & $f_{xx}>0$, f has a rel min @ (x_0,y_0)

b) If $D>0$ & $f_{xx}<0$, f has a rel max @ (x_0,y_0)

c) If $D<0$, f has a saddle point @ (x_0,y_0)

d) If $D=0$, no conclusion

P.1000

12) $f(x,y) = xy - x^3 - y^2$

$\frac{\partial f}{\partial x} = y - 3x^2$ $\quad$ $\frac{\partial f}{\partial y} = x - 2y$

$y - 3x^2 = 0$ $\quad$ $x - 2y = 0$

$y - 3(2y)^2 = 0$ $\quad$ $x = 2y$

$y - 12y^2 = 0$

$y(1-12y) = 0$

$y = 0,\ y = \frac{1}{12}$

$x = 0,\ x = \frac{1}{6}$

$\frac{\partial^2 f}{\partial x^2} = -6x$ $\quad$ $\frac{\partial^2 f}{\partial y^2} = -2$

$\frac{\partial^2 f}{\partial y \partial x} = 1$

critical point

$(0,0)$ & $(\frac{1}{6}, \frac{1}{12})$

$D = (-6x)(-2) - 1$

$D = 12x - 1$

$(0,0)$

$D = 12(0) - 1 = -1$ saddle point @ $(0,0)$

$(\frac{1}{6}, \frac{1}{12})$

$D = 12(\frac{1}{6}) - 1 = 1$

$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=\frac{1}{6}} = -1$ relative max @ $(\frac{1}{6}, \frac{1}{12})$

25a) $f(x,y) = 3xe^y - x^3 - e^{3y}$

$\frac{\partial f}{\partial x} = 3e^y - 3x^2$ $\frac{\partial f}{\partial y} = 3xe^y - 3e^{3y}$

$3e^y - 3x^2 = 0$ $3xe^y - 3e^{3y} = 0$

$e^y = x^2$ $xe^y - (e^y)^3 = 0$

$e^y = 1$ $x(x^2) - (x^2)^3 = 0$

$y = 0$ $x^3 - x^6 = 0$

$x^3(1 - x^3) = 0$

$\cancel{x = 0}$, $x = 1$

$\frac{\partial^2 f}{\partial x^2} = -6x$ $\frac{\partial^2 f}{\partial y^2} = 3xe^y - 9e^{3y}$

$\frac{\partial^2 f}{\partial y \partial x} = 3e^y$

crit @ $(1,0)$

$D = -6x(3xe^y - 9e^{3y}) - 3e^y$

$D = -6(1)(3(1)e^0 - 9e^0) - 3e^0$

$D = -6(3-9) - 3 = \underline{33}$

& $\frac{\partial^2 f}{\partial x^2} = -6$ ∴ rel max

b) $\lim_{x \to -\infty} f(x,0) = \lim_{x \to -\infty} (3x - x^3 - 1) = +\infty$

∴ no absolute max

30) $f(x,y) = xe^y - x^2 - e^y$; R: square w/ vertices of $(0,0), (0,1), (2,1), (2,0)$

$\frac{\partial f}{\partial x} = e^y - 2x$ $\frac{\partial f}{\partial y} = xe^y - e^y$

$e^y(x-1) = 0$

$e^y - 2 = 0$ $x = 1$

$e^y = 2$

$y = \ln 2$

critical pt: $(1, \ln 2)$

$x = 0$ $-e^y$ $\partial f: -e^y = 0$ no crit

$x = 2$ $2e^y - 4$ $\partial f: 2e^y = 0$ no crit

$y = 0$ $x - x^2 - 1$ $\partial f: 1 - 2x = 0$ $x = \frac{1}{2}$ $(\frac{1}{2}, 0)$

$y = 1$ $ex - x^2 - e$ $\partial f: e - 2x = 0$ $x = e/2$ $(e/2, 0)$

(x,y)	$f(x,y)$
$(0,0)$	-1
$(0,1)$	$-e$
$(2,1)$	$e-4$
$(2,0)$	-3
$(1,\ln 2)$	-1
$(\frac{1}{2},0)$	$-3/4$
$(e/2,0)$	$\frac{e}{2} - \frac{e^2}{4} - 1 \approx -1.488$

absolute max $-\frac{3}{4}$

absolute min -3

P.1000

34) Min $S = x^2+y^2+z^2$

subject to $x+y+z=27$

$x>0, y>0, z>0$

$z = 27-x-y$

$\therefore$

$S = x^2+y^2+(27-x-y)^2$

$S_x = 2x - 2(27-x-y)$

$S_y = 2y - 2(27-x-y)$

$$\begin{cases} 4x+2y-54=0 \\ 2x+4y-54=0 \end{cases}$$

$$\begin{cases} 2x+y = 27 \\ x+2y = 27 \end{cases}$$

$-3y = -27$

$\underline{y=9}$

$\therefore$

$2x+9 = 27$

$\underline{x = 9}$

so, critical pt is $(9,9)$

ck $D = (S_{xx})(S_{yy}) - (S_{xy})^2$

$D = (4)(4) - (2)^2$

$D = 12 > 0$

$S_{xx} > 0$

$\therefore$ $(9,9)$ is rel min

$z = 9$

$\therefore$ Three positive numbers whose sum is 27 and whose sum of squares is a minimum is

$x=9,\ y=9,\ z=9$

14.9

Two general problems to solve:

1) Two variable extremum w/ one constraint

2) Three variable extremum w/ one constraint

One way is to solve the constraint for one or two variable, and substitute into the fn under consideration. This method assumes you can solve the constraint.

In cases where the constraint can't be isolated, other methods must be applied.

If you graph the constraint fn and overlay the fn under consideration, you see points of possible maxima/minima. The point where the fn and the level curve of the constraint have a common normal line suggests that a max or min, if it exists, occurs at a point (x_0, y_0) on the constraint curve at which the gradient vectors ∇f & ∇g are scalar multiples of one another.

This scalar is called a Lagrange multiplier, λ (lambda)

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \qquad (4)$$

The Method of Lagrange multipliers for finding constrained relative extrema is to look for points on the constraint curve $g(x,y)=0$ @ which (4) is satisfied for some scalar λ

In 3-variable w/ one constraint, $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$

P.1008

8) $f(x,y) = x - 3y - 1$; $x^2 + 3y^2 = 16$

$\nabla f = 1\vec{i} - 3\vec{j} = \lambda(2x\vec{i} + 6y\vec{j})$

$1 = 2\lambda x \qquad -3 = 6\lambda y$

$x=0 \quad 1 \neq 0 \qquad y = 0 \quad -3 \neq 0$

so x & y are non zero

$\lambda = \frac{1}{2x} \qquad \lambda = -\frac{1}{2y}$

$\frac{1}{2x} = -\frac{1}{2y}$

$-x = y$

$y = -x \therefore x^2 + 3(-x)^2 = 16$

$4x^2 = 16$

$x^2 = 4$

$x = \pm 2 \qquad x = 2, y = -2$

$x = -2, y = 2$

$(2,-2)$ $\qquad$ $(-2,2)$

$f(2,-2) = 2+6-1 = 7 \qquad f(-2,2) = -2-6-1 = -9$

Abs max of 7 @ $(2,-2)$

Abs min of -9 @ $(-2,2)$

p.2 14.9

P. 1009

12) $f(x,y,z) = x^4+y^4+z^4$; $x^2+y^2+z^2=1$

$\nabla f = 4x^3\vec{i} + 4y^3\vec{j} + 4z^3\vec{k}$; $\nabla g = 2x\vec{i}+2y\vec{j}+2z\vec{k}$

$4x^3\vec{i} + 4y^3\vec{j} + 4z^3\vec{k} = \lambda(2x\vec{i}+2y\vec{j}+2z\vec{k})$

$4x^3 = 2x\lambda \qquad 4y^3 = 2y\lambda \qquad 4z^3 = 2z\lambda$

Case 1: $x, y, z \neq 0$

$\lambda = 2x^2 \qquad \lambda = 2y^2 \qquad \lambda = 2z^2$

$2x^2 = 2y^2 = 2z^2$

$x = \pm y = \pm z \Rightarrow x^2+x^2+x^2 = 1$

$3x^2 = 1$

$x = \pm\frac{1}{\sqrt{3}} \qquad f(x,y,z) = \frac{3}{9} = \frac{1}{3}$

Case 2: $x = 0$; $y, z \neq 0$

$\lambda = 2y^2 = 2z^2$

$y = \pm z. \qquad 0^2+y^2+y^2=1$

$2y^2 = 1$

$y = \pm\frac{1}{\sqrt{2}} \qquad f(x,y,z) = \frac{2}{4} = \frac{1}{2}$

Case 3: $x = y = 0$, $z \neq 0$

$z^2 = 1$

$z = \pm 1 \qquad f(x,y,z) = 1$

Absolute min @ $(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$ of $\frac{1}{3}$

Absolute max @ $(0,0,\pm 1)$ of 1

14) point on $y = 2x+3$ closest to $(4,2)$

$f(x,y) = (x-4)^2+(y-2)^2$; $g(x,y) = y-2x-3$

$\nabla f = 2(x-4)\vec{i} + 2(y-2)\vec{j}$; $\nabla g = -2\vec{i} + 1\vec{j}$

$2(x-4) = -2\lambda \qquad 2(y-2) = 1\lambda$

$-x+4 = \lambda \qquad 2y-4 = \lambda$

$-x+4 = 2y-4$

$x = -2y+8$

$y = 2(-2y+8)+3 \qquad x = -2\left(\frac{19}{5}\right)+8$

$y = -4y+16+3$

$5y = 19 \qquad x = -\frac{38}{5}+8$

$y = \frac{19}{5} \qquad x = \frac{2}{5}$

$\therefore$ the point is $\left(\frac{2}{5}, \frac{19}{5}\right)$

20) $T(x,y) = 4x^2-4xy+y^2$; $g(x,y) = x^2+y^2 = 25$

$\nabla T = (8x-4y)\vec{i} + (-4x+2y)\vec{j}$; $\lambda\nabla g = 2x\lambda\vec{i} + 2y\lambda\vec{j}$

$8x-4y = 2x\lambda \qquad -4x+2y = 2y\lambda$

$\frac{4x-2y}{x} = \lambda \qquad \frac{-2x+y}{y} = \lambda$

Note: if $x=0, y=0$ $0 \neq 25$ so $x, y \neq 0$

$\frac{4x-2y}{x} = \frac{-2x+y}{y}$

$4xy-2y^2 = -2x^2+xy$

$2x^2+3xy-2y^2 = 0$

$(2x-y)(x+2y) = 0$

$y = 2x \qquad x = 2y$

$x^2+4x^2 = 25 \qquad 4y^2+y^2 = 25$

$5x^2 = 25 \qquad 5y^2 = 25$

$x^2 = 5 \qquad y^2 = 5$

$x = \pm\sqrt{5} \qquad y = \pm\sqrt{5}$

$y = \pm 2\sqrt{5} \qquad x = \pm 2\sqrt{5}$

(x,y)	$T(x,y)$
$(\sqrt{5}, 2\sqrt{5})$	0
$(\sqrt{5}, -2\sqrt{5})$	80
$(-\sqrt{5}, 2\sqrt{5})$	80
$(-\sqrt{5}, -2\sqrt{5})$	0
$(2\sqrt{5}, \sqrt{5})$	45
$(-2\sqrt{5}, \sqrt{5})$	125
$(2\sqrt{5}, -\sqrt{5})$	125
$(-2\sqrt{5}, -\sqrt{5})$	45

Highest temp: 125°

Lowest temp: 0°

25) Minimize $f(x,y,z) = 10(2xy)+5(2xz+2yz)$

constraint $g(x,y,z) = xyz-16 = 0$

$\nabla f = (20y+10z)\vec{i} + (20x+10z)\vec{j} + (10x+10y)\vec{k}$

$\lambda\nabla g = \lambda yz\vec{i} + \lambda xz\vec{j} + \lambda xy\vec{k} \qquad x,y,z \neq 0$

$20y+10z = \lambda yz \qquad 20x+10z = \lambda xz \qquad 10x+10y = \lambda xy$

$20+10\frac{z}{y} = \lambda z \qquad 20+10\frac{z}{x} = \lambda z \qquad 20x = \lambda x^2$

$20 = \lambda x$

$20+10\frac{z}{y} = 20+10\frac{z}{x}$

$x = y \qquad 20x+10z = 20z$

$20x = 10z$

$2x = z$

$V = x(x)(2x) = 16 \rightarrow x = 2\text{ ft}, y = 2\text{ ft}, z = 4\text{ ft}$

$2x^3 = 16 \qquad 8f(x,y,z) = 240$ cents

$x^3 = 8 \qquad$ or \$2.40

$x = 2$