Ch. 15 Multiple Integrals

15.1 Double Integration

The reverse of partial differentiation is partial integration

$\int_a^b f(x,y)\,dx$ & $\int_c^d f(x,y)\,dy$ are partial definite integrals

You hold one variable constant as you integrate wrt the indicated variable

Integrated Integration

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

or $$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

P.1019

5) $\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y}\,dy\,dx$

$\int_0^{\ln 3}\left[e^{x+y} \Big|_0^{\ln 2}\right] dx$

$\int_0^{\ln 3}\left[e^{x+\ln 2} - e^{x}\right] dx$

$\int_0^{\ln 3}\left[2e^{x} - e^{x}\right] dx$

$\int_0^{\ln 3} e^{x}\,dx = e^{x}\Big|_0^{\ln 3}$

$3 - 1 = \underline{\underline{2}}$

9) $\int_0^1 \int_0^1 \frac{x}{(xy+1)^2}\,dy\,dx$ $\quad u = xy+1$, $du = x\,dy$

$\int_0^1\left[u^{-2}\,du\right] dx$

$\int_0^1\left[-\frac{1}{(xy+1)}\Big|_0^1\right] dx$

$\int_0^1\left[-\frac{1}{x+1} + 1\right] dx$ $\quad u = x+1$, $du = dx$

$\int \left[-\frac{1}{u}\right] du + 1\,dx$

$-\ln|x+1|\Big|_0^1 + x\Big|_0^1$

$-\ln 2 + 0 + 1 = \underline{\underline{1 - \ln 2}}$

13) $\iint_R 4xy^3\,dA$; $R = \{(x,y) \mid -1 \le x \le 1, -2 \le y \le 2\}$

$\int_{-1}^1 \int_{-2}^2 4xy^3\,dy\,dx = \int_{-1}^1 \left[xy^4\right]_{-2}^2 dx$

$= \int_{-1}^1 0\,dx = \underline{\underline{0}}$

20) $V = \int_1^3 \int_0^2 (3x^3 + 3x^2y)\,dy\,dx$

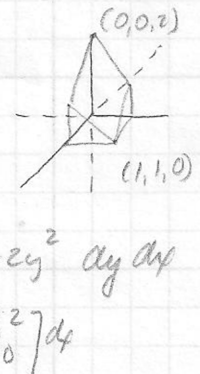
$\int_1^3\left[\left(3x^3y + \frac{3}{2}x^2y^2\right)\Big|_0^2\right] dx$

$\int_1^3 (6x^3 + 6x^2)\,dx$

$\frac{3}{2}x^4 + 2x^3\Big|_1^3 = \frac{243}{2} + \frac{54}{1} - \frac{3}{2} - 2$

$= \frac{240}{2} + 52 = \frac{344}{2} = \underline{\underline{172}}$

24a) $V = \int_0^1 \int_0^1 (2 - x - y)\,dy\,dx$



29) $f_{av} = \frac{1}{2}\int_0^1 \int_0^2 10 - 8x^2 - 2y^2\,dy\,dx$

$= \frac{1}{2}\int_0^1\left[\left(10y - 8x^2y - \frac{2}{3}y^3\right)\Big|_0^2\right] dx$

$= \frac{1}{2}\int_0^1\left(20 - 16x^2 - \frac{16}{3}\right) dx = \frac{1}{2}\left[20x - \frac{16}{3}x^3 - \frac{16}{3}x\right]\Big|_0^1$

$= \frac{1}{2}\left[20 - \frac{16}{3} - \frac{16}{3}\right] = \underline{\underline{\frac{14}{3}}}$ °C

15.2

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx \quad \text{or} \quad \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Type I Region: bounded left-right by vertical line $x = a$, $x = b$;
top-bottom by continuous curves $y = g_2(x)$, $y = g_1(x)$

Type II Region: bounded top-bottom by horizontal lines $y = d$, $y = c$;
left-right by continuous curves $x = h_1(y)$, $x = h_2(y)$

To reverse the order of integration, DRAW THE REGION FIRST.

P. 1027

4) $\int_{1/4}^{1} \int_{x^2}^{x} \sqrt{x}\, y^{-\frac{1}{2}}\, dy\, dx$

$\int_{1/4}^{1} 2\sqrt{xy} \Big|_{x^2}^{x}\, dx$

$\int_{1/4}^{1} 2x - 2x^{3/2}\, dx$

$x^2 - \frac{4}{5}x^{5/2} \Big|_{1/4}^{1}$

$\left(1 - \frac{4}{5}\right) - \left(\frac{1}{16} - \frac{1}{40}\right)$

$\frac{80 - 64 - 5 + 2}{80} = \frac{13}{80}$

10) $\int_1^2 \int_0^{y^2} e^{\frac{x}{y^2}}\, dx\, dy$

$\int_1^2 y^2 e^{\frac{x}{y^2}} \Big|_0^{y^2}\, dy$

$\int_1^2 ey^2 - y^2\, dy$

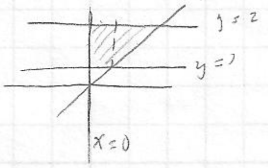
$\int_1^2 y^2(e-1)\, dy$

$\frac{1}{3}y^3(e-1) \Big|_1^2$

$\frac{8}{3}(e-1) - \frac{1}{3}(e-1)$

$\frac{7}{3}(e-1)$

14) $\iint_R xy^2\, dA \quad R: y = 1, y = 2, x = 0, x = y$



Type 1: $\int_0^1 \int_1^2 xy^2\, dy\, dx$

$+ \int_1^2 \int_x^2 xy^2\, dy\, dx$

$\int_0^1 \frac{1}{3}xy^3 \Big|_1^2\, dx + \int_1^2 \frac{1}{3}xy^3 \Big|_x^2\, dx$

$\int_0^1 \frac{7}{3}x\, dx + \int_1^2 \frac{8}{3}x - \frac{1}{3}x^4\, dx$

$\frac{7}{6}x^2 \Big|_0^1 + \left(\frac{4}{3}x^2 - \frac{1}{15}x^5\right)\Big|_1^2$

$\frac{7}{6} + \frac{16}{3} - \frac{32}{15} - \frac{4}{3} + \frac{1}{15}$

$\frac{35 + 160 - 64 - 40 + 2}{30} = \frac{93}{30} = \frac{31}{10}$

Type 2: $\int_1^2 \int_0^y xy^2\, dx\, dy$

$\int_1^2 \frac{1}{2}x^2y^2 \Big|_0^y\, dy$

$\int_1^2 \frac{1}{2}y^4\, dy$

$\frac{1}{10}y^5 \Big|_1^2$

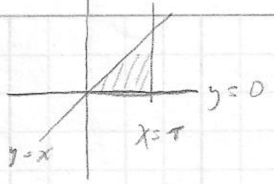
$\frac{1}{10}(32-1)$

$\frac{31}{10}$

p.2 15.2

P. 1027

18) $\iint_R x\cos y \, dA$; R: $y=x,\ y=0,\ x=\pi$



$\int_0^{\pi}\int_0^{x} x\cos y\, dy\, dx$

$\int_0^{\pi} x\sin y\Big|_0^{x} dx \Rightarrow \int_0^{\pi} x\sin x\ dx$

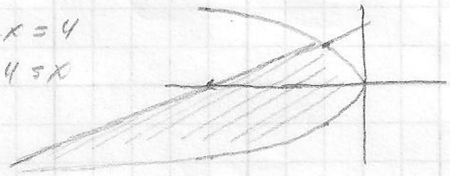
	u	dv
+	x	sin x
−	1	−cos x
+	0	−sin x

$-x\cos x+\sin x\Big|_0^{\pi}$

π

26) $y^2=-x$, $3y-x=4$

$-y^2=x$ $\quad 3y-4=x$



$-y^2=3y-4$

$0=y^2+3y-4$

$0=(y+4)(y-1)$

$y=-4,\ y=1$

$\int_{-4}^{1}\int_{3y-4}^{-y^2} dx\, dy$

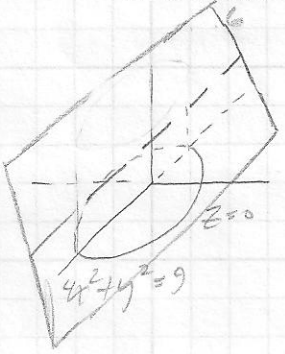
$\int_{-4}^{1} x\Big|_{3y-4}^{-y^2} \Rightarrow \int_{-4}^{1}(-y^2-3y+4)\, dy$

$= -\frac{1}{3}y^3-\frac{3}{2}y^2+4y\Big|_{-4}^{1}$

$=\left(-\frac{1}{3}-\frac{3}{2}-4\right)-\left(\frac{64}{3}-\frac{48}{2}-16\right)$

$=-\frac{65}{3}+\frac{45}{2}+20=\frac{-130+135+120}{6}=\frac{125}{6}$

35) $4x^2+y^2=9,\ z=0,\ z=y+3$

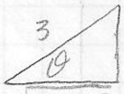


$\int_{-3/2}^{3/2}\int_{-\sqrt{9-4x^2}}^{\sqrt{9-4x^2}}[(y+3)-0]\, dy\, dx$

$\int_{-3/2}^{3/2}\frac{1}{2}y^2+3y\Big|_{-\sqrt{9-4x^2}}^{\sqrt{9-4x^2}} dx$

$\int_{-3/2}^{3/2}\cancel{\frac{9-4x^2}{2}}+3\sqrt{9-4x^2}-\left(\cancel{\frac{9-4x^2}{2}}-3\sqrt{9-4x^2}\right)dx = 6\int_{-3/2}^{3/2}\sqrt{9-4x^2}\ dx$

$12\int_0^{3/2}\sqrt{9-4x^2}\ dx$



$3\cos\theta=\sqrt{9-4x^2}$

$\frac{3}{2}\sin\theta=x$

$\frac{3}{2}\cos\theta\, d\theta=dx$

$12\int 3\cos\theta\cdot\frac{3}{2}\cos\theta\, d\theta$

$54\int\frac{1}{2}(1+\cos 2\theta)\, d\theta = 27\left[\theta+\frac{1}{2}\sin 2\theta\right]=27\left[\theta+\sin\theta\cos\theta\right]$

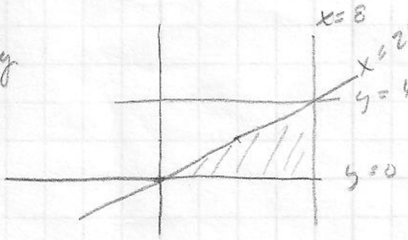
$=27\left[\sin^{-1}\frac{2}{3}x+\frac{2}{3}x\frac{\sqrt{9-4x^2}}{3}\right]\Big|_0^{3/2}=27\left[\left(\frac{\pi}{2}+0\right)-(0)\right]=\frac{27\pi}{2}$

p.3 15.2

P.1028

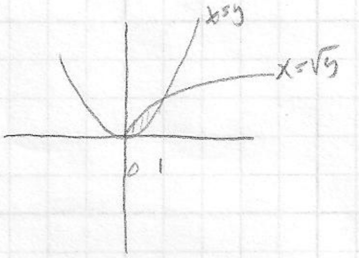
42) $\int_{y=0}^{y=4}\int_{x=2y}^{x=8} f(x,y)\,dx\,dy$

$\int_0^8\int_0^{\frac{1}{2}x} f(x,y)\,dy\,dx$

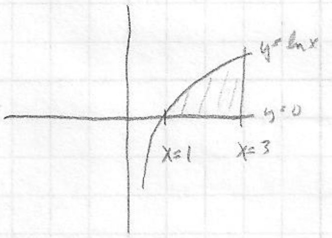


46) $\int_{y=0}^{y=1}\int_{x=y^2}^{x=\sqrt{y}} f(x,y)\,dx\,dy$

$\int_0^1\int_{x^2}^{\sqrt{x}} f(x,y)\,dy\,dx$



50) $\int_1^3\int_0^{\ln x} x\,dy\,dx$



$\int_0^{\ln 3}\int_{e^y}^{3} x\,dx\,dy$

$\int_0^{\ln 3} \frac{1}{2}x^2\Big|_{e^y}^{3}\,dy$

$\int_0^{\ln 3}\left(\frac{9}{2}-\frac{1}{2}e^{2y}\right)dy$

$\frac{9}{2}y-\frac{1}{4}e^{2y}\Big|_0^{\ln 3}$

$\frac{9}{2}\ln 3-\frac{1}{4}(9)-0+\frac{1}{4}$

$\frac{9}{2}\ln 3-\frac{8}{4}=\frac{9}{2}\ln 3-2$

58) $f(x,y)=x^2-xy$

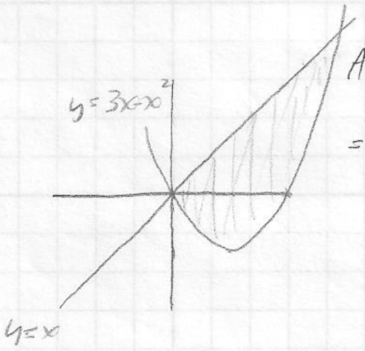
$y=x \quad y=3x-x^2$

$x=3x-x^2$

$x^2-2x=0$

$x(x-2)=0$

$x=0,\ x=2$



$A=\int_0^2 3x-x^2-x\,dx$

$=\frac{4}{3}$

$(3x-x^2)^2=9x^2-6x^3+x^4$

$\frac{1}{\frac{4}{3}}\int_0^2\int_x^{3x-x^2} x^2-xy\,dy\,dx \Rightarrow f_{av}=\frac{3}{4}\int_0^2 x^2y-\frac{1}{2}xy^2\Big|_x^{3x-x^2}\,dx$

$f_{av}=\frac{3}{4}\int_0^2 \cancel{3x^3}-x^4-\cancel{\frac{9}{2}x^3}+\cancel{3}x^4-\frac{1}{2}x^5\cancel{-x^3}+\cancel{\frac{1}{2}x^3}\,dx \Rightarrow f_{av}=\frac{3}{4}\int_0^2 -2x^3+2x^4-\frac{1}{2}x^5\,dx$

$f_{av}=\frac{3}{4}\left[-\frac{2}{4}x^4+\frac{2}{5}x^5-\frac{1}{12}x^6\right]_0^2=\frac{3}{4}\left[-8+\frac{64}{5}-\frac{\overset{32}{\cancel{128}}}{\cancel{12}_3}\right]=\frac{3}{4}\left(\frac{-120+192-160}{15}\right)=\frac{3}{4}\left(-\frac{88}{15}\right)=-\frac{22}{5}$

15.3

Volume in Polar Coordinates

Given a fn $f(r,\theta)$ that is continuous & nonnegative on a simple polar region R, find the volume of the solid enclosed between the region R and the surface whose eqn in cylindrical coordinates is $z = f(r,\theta)$

Polar doubles are also known as double integrals in Polar coordinates to distinguish from double integrals in rectangular coor.

Thm 15.3.3

If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ & $\theta = \beta$ and the curves $r = r_1(\theta)$ & $r = r_2(\theta)$ & if $f(r,\theta)$ is continuous on R, then

$$\iint_R f(r,\theta)\, dA = \int_\alpha^\beta \int_{r(\theta_1)}^{r(\theta_2)} f(r,\theta)\, r\, dr\, d\theta$$

P.1034

6) $\int_0^{\pi/2}\int_0^{\cos\theta} r^3\, dr\, d\theta$

$\int_0^{\pi/2} \frac{1}{4} r^4 \Big|_0^{\cos\theta} d\theta$

$\int_0^{\pi/2} \frac{1}{4}\cos^4\theta \; d\theta \quad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

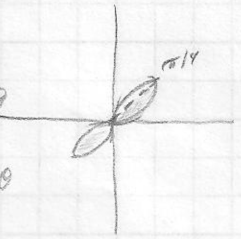
$\frac{1}{4}\int_0^{\pi/2} \left[\frac{1}{2}(1+\cos 2\theta)\right]^2$

$\frac{1}{16}\int_0^{\pi/2} \left[1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right] d\theta$

$\frac{1}{16}\left[\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]\Big|_0^{\pi/2}$

$\frac{1}{16}\left[\frac{3\pi}{4}\right] = \frac{3\pi}{64}$

8) $r = \sin 2\theta$



$A = 4\int_0^{\pi/4}\int_0^{\sin 2\theta} r\, dr\, d\theta$

$= 4\int_0^{\pi/4} \frac{1}{2}\sin^2 2\theta\; d\theta$

$= 2\int_0^{\pi/4} \frac{1}{2}\left[1 - \cos 2\theta\right] d\theta$

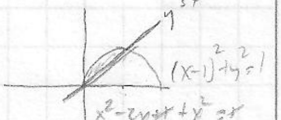
$= \int_0^{\pi/4} (1 - \cos 2\theta)\, d\theta \quad \begin{matrix} u = 2\theta \\ du = 2\, d\theta \end{matrix}$

$= \left[\theta - \frac{1}{2}\sin 2\theta\right]\Big|_0^{\pi/4}$

$= \left[\frac{\pi}{4} - \frac{1}{2}\right]$

$= \frac{2\pi - 1}{8}$

22) $\int_0^1 \int_x^{\sqrt{1-(x-1)^2}} 2y\, dA \qquad \begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}$



$x^2 - 2x + 1 + x^2 = 1$

$2x(x-1) = 0$

$x = 0 \quad x = 1$

$\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} 2r\sin\theta\; r\, dr\, d\theta$

$\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} 2r^2\sin\theta\; dr\, d\theta$

$\int_{\pi/4}^{\pi/2} \frac{2}{3} r^3 \sin\theta \Big|_0^{2\cos\theta} d\theta$

$\frac{2}{3}\int_{\pi/4}^{\pi/2} 8\cos^3\theta\, \sin\theta\, d\theta \qquad \begin{matrix} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{matrix}$

$-\frac{16}{3}\int u^3 \cdot du$

$-\frac{16}{3}\cdot\frac{1}{4}\cos^4\theta \Big|_{\pi/4}^{\pi/2}$

$-\frac{4}{3}\left[0 - \frac{1}{4}\right] = \frac{1}{3}$

p.215.4

P.1045

6b)

$y - 2z = 5$ between $x=0,\ x=3$

$x = u,\ y = 5+2z,\ z = v,\ 0 \le u \le 3$

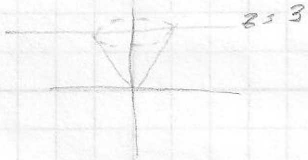
$y = 5 + 2v$

10) $z = e^{-(x^2+y^2)}$ $\quad x = r\cos\theta,\ y = r\sin\theta \Rightarrow r^2 = x^2+y^2$

$\therefore z = e^{-r^2}$

14) $z = \sqrt{x^2+y^2},\ z = 3$ $\quad x = r\cos\theta,\ y = r\sin\theta \Rightarrow r^2 = x^2+y^2$

$z = r,\ r \le 3$



22) $r(u,v) = \sin u \cos v\,\vec{i} + 2\sin u \sin v\,\vec{j} + 3\cos u\,\vec{k};\ 0 \le u \le \pi,\ 0 \le v < 2\pi$

$x = \sin u \cos v,\ y = 2\sin u \sin v,\ z = 3\cos u$

$x^2 = \sin^2 u \cos^2 v,\ \left(\frac{y}{2}\right)^2 = \sin^2 u \sin^2 v,\ \left(\frac{z}{3}\right)^2 = \cos^2 u$

$x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2 = \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u$

$x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2 = \sin^2 u(\cos^2 v + \sin^2 v) + \cos^2 u$

$x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^3 = \sin^2 u + \cos^2 u$

$x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2 = 1$, ellipsoid

15.4

Surface in 3-space can be represented by three equations involving two parameter,

$x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ ①

Note: When you convert into spherical coordinates, the constant ϕ-curves are called lines of latitude and the constant θ-curves are called lines of longitude

The parametric eqns mentioned earlier can be expressed in vector form as

$\vec{r} = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$ ②

Here $\vec{r}(u,v)$ is a vector-valued fn of two variables

15.4.1 (defn)

If a parametric surface σ is the graph of $\vec{r} = \vec{r}(u,v)$, and if $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \neq \vec{0}$ at some point on the surface, then the principal unit normal vector to the surface at that point is denoted by $\vec{n}$ or $\vec{n}(u,v)$ and is defined as

$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\|}$$

We will say that σ is a smooth parametric surface on a region R of the uv-plane if $\frac{\partial \vec{r}}{\partial u}$ & $\frac{\partial \vec{r}}{\partial v}$ are continuous on R and $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \neq \vec{0}$ on R.

Surface Area $SA = \iint_R \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| dA$ [$\vec{r}$ defined in ②] ③

Surface Area of surfaces of the form $z = f(x,y)$

①, ②, & ③ are modified as follows

$x = u$, $y = u$, $z = f(u,v)$

$\vec{r} = u\vec{i} + u\vec{j} + f(u,v)\vec{k}$

$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = -\frac{\partial z}{\partial x}\vec{i} - \frac{\partial z}{\partial y}\vec{j} + \vec{k}$

$\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$

$\therefore SA = \int_R \int \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$

p3 15.4

31) $x = 3v \sin u,\ y = 2v \cos u,\ z = u^2\ ;\ (0,2,0)$

$\vec{r}(u,v) = 3v \sin u\,\vec{i} + 2v \cos u\,\vec{j} + u^2\,\vec{k}$

$\frac{\partial \vec{r}}{\partial u} = 3v \cos u\,\vec{i} - 2v \sin u\,\vec{j} + 2u\,\vec{k}$ $\quad \frac{\partial \vec{r}}{\partial v} = 3 \sin u\,\vec{i} + 2 \cos u\,\vec{j} + 0\vec{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & k \\ 3v\cos u & -2v\sin u & 2u \\ 3\sin u & 2\cos u & 0 \end{vmatrix} = -4u\cos u\,\vec{i} + 6u \sin u\,\vec{j} + 6v\,\vec{k} \Rightarrow$$

$u = 0,\ v = 1 \quad \underline{\underline{r_u \times r_v = 6\vec{k}}},\ z = 0$

46) $x^2 + y^2 + z^2 = 8$ mid $z = \sqrt{x^2+y^2}$

$$S = \int\int \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA$$

$\frac{\partial z}{\partial x} : 2x + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}$

$\frac{\partial z}{\partial y} : 2y + 2z\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$

$\sqrt{(\quad) + (\quad) + 1} \Rightarrow \sqrt{\left(-\frac{x}{z}\right)^2 + \left(-\frac{y}{z}\right)^2 + 1} \Rightarrow \frac{x^2+y^2+z^2}{z^2} = \frac{8}{8-x^2-y^2}$

$z^2 = 8 - x^2 - y^2$ $\quad \Rightarrow \frac{r^2+z^2}{z^2} = \frac{8}{8-r^2}$

$z =$

15.5

A single integral of a fn $f(x)$ is defined over a finite closed interval on the x-axis

A double integral of a fn $f(x,y)$ is defined over a finite closed region R in the xy-plane.

A triple integral of a fn $f(x,y,z)$ is defined over a closed solid region G in the xyz-coordinate system.

Thm 15.5.1

Let G be a rectangular box defined by

$a \le x \le b,\; c \le y \le d,\; k \le z \le l$

$$\iiint_G f(x,y,z)\,dV = \int_a^b \int_c^d \int_k^d f(x,y,z)\,dz\,dy\,dx$$

or any other of the 5 equivalent ∫ with the order changed.

Thm 15.5.2

Let G be a simple xy-solid w/upper surface $z = g_2(x,y)$ & lower surface $z = g_1(x,y)$, and let R be the projection of G on the xy-plane. If $f(x,y,z)$ is continuous on G, then

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right] dA \qquad (3)$$

The limits of integration can be obtained by:

① Find an eq'n $z = g_2(x,y)$ for the upper surface & an eq'n $z = g_1(x,y)$ for the lower surface of G. g_1 & g_2 determine your lower and upper limits of integration.

② Make a 2D sketch of the projection R of the solid on the xy-plane. From this sketch determine the limits of integration for the double integral over R in (3).

Volume of G = $\iiint_G dV$

p. 2 15.5

Integration in other orders:

for a xz-solid (parallel to y-axis) it is usually best to integrate wrt y first, and for a yz-solid (parallel to x-axis) it is usually best to integrate wrt x first.

P. 1054

7) $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-5+x^2+y^2}^{3-x^2-y^2} x \, dz \, dy \, dx$

$\int_0^2 \int_0^{\sqrt{4-x2}} xz \Big|_{-5+x^2+y^2}^{3-x^2-y^2} dy \, dx$

$\int_0^2 \int_0^{\sqrt{4-x2}} 3x - x^3 - xy^2 + 5x - x^3 - xy^2 \, dy \, dx$

$\int_0^2 \int_0^{\sqrt{4-x^2}} (8x - 2x^3 - 2xy^2) \, dy \, dx$

$\int_0^2 \int_0^{\sqrt{4-x^2}} \left[2x(4-x^2) - 2xy^2\right] dy \, dx$

$\int_0^2 \left[2x(4-x^2)y - \frac{2}{3}xy^3\right]\Big|_0^{\sqrt{4-x2}} dx$

$\int_0^2 \left[2x(4-x^2)^{3/2} - \frac{2}{3}x(4-x^2)^{3/2}\right] dx$

$\frac{4}{3}\int_0^2 x(4-x^2)^{3/2} \, dx$ $\quad u = 4-x^2$, $du = -2x \, dx$

$-\frac{2}{3}\int u^{3/2} \, du$

$-\frac{2}{3} \cdot \frac{2}{5}(4-x^2)^{5/2} \Big|_0^2$

$-\frac{4}{15}(0 - 32)$

$\frac{128}{15}$

p3 15.5

P. 1054

17) $y = x^2,\ y + z = 4,\ z = 0$

$$2\int_0^2 \int_{x^2}^4 \int_0^{4-y} dz\, dy\, dx$$

$$2\int_0^2 \int_{x^2}^4 z \Big|_0^{4-y} dy\, dx$$

$$2\int_0^2 \int_{x^2}^4 (4-y)\, dy\, dx$$

$$2\int_0^2 \left(4y - \frac{1}{2}y^2\right)\Big|_{x^2}^4$$

$$2\int_0^2 \left(8 - 4x^2 + \frac{1}{2}x^4\right) dx$$

$$\int_0^2 \left(16 - 8x^2 + x^4\right) dx$$

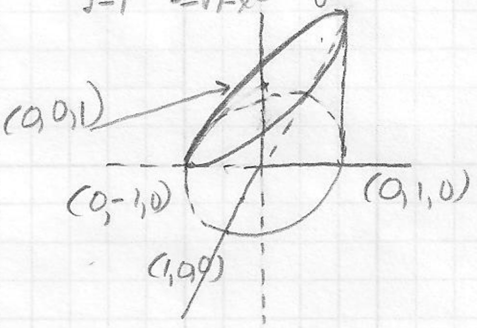
$$\left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^2$$

$$32 - \frac{64}{3} + \frac{32}{5}$$

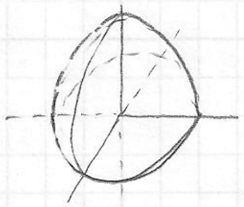
$$480 - 320 + 96$$

$$\underline{\underline{\frac{256}{15}}}$$

23a) $$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{y+1} dz\, dy\, dx$$



32a) $$\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{9-y^2-z^2}} f(x,y,z)\, dx\, dy\, dz$$



$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} f(x,y,z)\, dz\, dy\, dx$$

Ch.15

15.6

An idealized flat object that is thin enough to be viewed as a two-dimensional plane region is called a lamina.

A lamina is called homogeneous if its composition is uniform throughout and it is called inhomogeneous otherwise.

The density of a homogeneous lamina of mass M & area A is $\delta = \frac{M}{A}$

For inhomogeneous lamina, if a lamina w/ a continuous density fn $\delta(x,y)$ occupies a region R in the xy-plane, then its total mass M is given by

$$M = \iint_R \delta(x,y)\, dA$$

If a point-mass m is located on an axis at x, then the tendency for that mass to produce a rotation of the beam @ a point a on the axis is measured by the moment of m about $x=a$

moment of m @ $x=a$ $= m(x-a)$ where $x-a$ is called the lever arm.

Center of Gravity $(\bar{x}, \bar{y})$ of a Lamina

$$\bar{x} = \frac{\iint_R x\,\delta(x,y)\, dA}{\iint_R \delta(x,y)\, dA}, \quad \bar{y} = \frac{\iint_R y\,\delta(x,y)\, dA}{\iint_R \delta(x,y)\, dA} \qquad (7-8)$$

The numerator for $\bar{x}$ is denoted M_y, the 1st moment of the lamina @ y-axis

The numerator for $\bar{y}$ is denoted M_x, the 1st moment of the lamina @ x-axis

These reduce formulas (7-8) to

$$\bar{x} = \frac{M_y}{M} = \frac{1}{\text{mass of } R} \iint_R x\,\delta(x,y)\, dA \qquad (9)$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{\text{mass of } R} \iint_R y\,\delta(x,y)\, dA \qquad (10)$$

note: $M_y = \iint_R x\,\delta(x,y)\, dA$, $M_x = \iint_R y\,\delta(x,y)\, dA$

p 2 15.6

In the special case of a homogeneous lamina the center of gravity is called the centroid of the lamina or the centroid of the region R.

Since δ is constant, it can be moved through the integral sign and cancelled, $\therefore$

$$\bar{x} = \frac{\iint_R x\, dA}{\iint_R dA} = \frac{1}{\text{area of } R}\iint_R x\, dA \; ; \; \bar{y} = \frac{\iint_R y\, dA}{\iint_R dA} = \frac{1}{\text{area of } R}\iint_R y\, dA$$

For a 3-dimensional solid G, the formulas for moments, center of gravity and centroid are similar to those for laminas.

If G is homogeneous, density is mass per unit volume $\delta = \frac{M}{V}$

If G is inhomogeneous and is in an xyz-coordinate system, then its density @ a general point (x, y, z) is specified by a density fn $\delta(x, y, z)$

mass of G = $M = \iiint_G \delta(x, y, z)\, dV$

Center of Gravity $(\bar{x}, \bar{y}, \bar{z})$ of a solid G (inhomogeneous solid)

$$\bar{x} = \frac{1}{M}\iiint_G x\,\delta(x,y,z)\, dV \; ; \; \bar{y} = \frac{1}{M}\iiint_G y\,\delta(x,y,z)\, dV \; ; \; \bar{z} = \frac{1}{M}\iiint_G z\,\delta(x,y,z)\, dV$$

Centroid $(\bar{x}, \bar{y}, \bar{z})$ of a solid G (homogeneous solid)

$$\bar{x} = \frac{1}{V}\iiint_G x\, dV \; ; \; \bar{y} = \frac{1}{V}\iiint_G y\, dV \; ; \; \bar{z} = \frac{1}{V}\iiint_G z\, dV$$

Theorem of Pappas

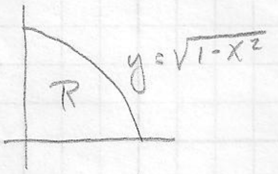
If R is a bounded plane region and L is a line that lies in the plane of R but is entirely on one side of R, then the volume of the solid formed by revolving R @ L is given by

volume = (area of R)(distance traveled by centroid)

p.3 15.6

p.1063

8)



$\bar{x} = \frac{1}{\text{area}} \iint_R x\,dA = \frac{4}{\pi}\int_0^1 \int_0^{\sqrt{1-x^2}} x\,dy\,dx = \frac{4}{\pi}\int_0^1 x\sqrt{1-x^2}\,dx$

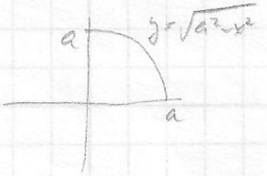
$= -\frac{2}{\pi}\int u^{1/2}\,du$

$= -\frac{2}{\pi}\cdot\frac{2}{3}(1-x^2)^{3/2}\Big|_0^1 = \frac{4}{3\pi}$

since this is a symmetric region, $\bar{y} = \frac{4}{3\pi}$

∴ Centroid $\left(\frac{4}{3\pi}, \frac{4}{3\pi}\right)$

15) $\delta(x,y) = xy$; 1st quad & bounded by $x^2+y^2=a^2$ & coor. axes



$m = \iint_R \delta(x,y)\,dA \Rightarrow m = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy\,dy\,dx$

$= \int_0^a \frac{1}{2}xy^2 \Big|_0^{\sqrt{a^2-x^2}} dx$

$= \frac{1}{2}\int_0^a x(a^2-x^2)\,dx$

$= \frac{1}{2}\int_0^a a^2x - x^3\,dx$

$= \frac{1}{2}\left[\frac{1}{2}a^2x^2 - \frac{1}{4}x^4\right]_0^a$

$\bar{x} = \bar{y}$ because of δ & symmetry of R, etc

$= \frac{1}{2}\left[\frac{1}{2}a^4 - \frac{1}{4}a^4\right] = \frac{1}{2}\left[\frac{1}{4}a^4\right] = \frac{1}{8}a^4$

$M_y = \int_0^a \int_0^{\sqrt{a^2-x^2}} x(xy)\,dy\,dx = \frac{1}{2}\int_0^a x^2y^2\Big|_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a (a^2x^2 - x^4)\,dx = \frac{1}{2}\left[\frac{1}{3}a^2x^3 - \frac{1}{5}x^5\right]_0^a$

$= \frac{1}{2}\left[\frac{1}{3}a^5 - \frac{1}{5}a^5\right] = \frac{1}{15}a^5$

$\bar{y} = \bar{x} = \frac{M_y}{m} = \frac{\frac{1}{15}a^5}{\frac{1}{8}a^4} = \frac{8a}{15}$ ∴ mass $= \frac{1}{8}a^4$, center of gravity $= \left(\frac{8a}{15}, \frac{8a}{15}\right)$

p.4 15.6

24) xy-plane & hemisphere $z = \sqrt{a^2-x^2-y^2}$

$\bar{x} = \bar{y} = 0$ from symmetry

$V = \frac{2\pi a^3}{3}$

$$\bar{z} = \frac{1}{V}\int\int\int z \, dV$$

$$\bar{z} = \frac{3}{2\pi a^3}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{\sqrt{a^2-x^2-y^2}} z \, dz \, dy \, dx$$

$$\bar{z} = \frac{3}{4\pi a^3}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z^2 \Big|_0^{\sqrt{a^2-x^2-y^2}} dy \, dx$$

$$\bar{z} = \frac{3}{4\pi a^3}\int_{-a}^{a}\int_{-\sqrt{a+x^2}}^{\sqrt{a^2-x^2}} (a^2-x^2-y^2) \, dy \, dx$$

$$\bar{z} = \frac{3}{4\pi a^3}\int_{0}^{2\pi}\int_{0}^{a} (a^2-r^2) \, r \, dr \, d\theta$$

$$\bar{z} = \frac{3}{4\pi a^3}\int_{0}^{2\pi}\int_{0}^{a} (a^2r-r^3) \, dr \, d\theta$$

$$\bar{z} = \frac{3}{4\pi a^3}\int_{0}^{2\pi} \left(\frac{a^2r^2}{2} - \frac{1}{4}r^4\right)\Big|_0^a \, d\theta$$

$$\bar{z} = \frac{3}{4\pi a^3}\int_{0}^{2\pi} \frac{1}{4}a^4 \, d\theta$$

$$\bar{z} = \frac{3}{4\pi a^3}\left[\frac{1}{4}a^4\theta\right]\Big|_0^{2\pi} = \frac{3a}{8} \quad \therefore \quad \underline{\underline{\left(0, 0, \frac{3a}{8}\right)}}$$

28) $f(x,y,z) = xz$, $y = 9-x^2 \ (x \geq 0)$, $x=0$, $y=0$, $z=0$, $z=1$

$$\text{Mass} = \int_0^3\int_0^{9-x^2}\int_0^1 xz \, dz \, dy \, dx$$

$$m = \int_0^3\int_0^{9-x^2} \frac{1}{2}xz^2\Big|_0^1 \, dy \, dx \Rightarrow m = \int_0^3\int_0^{9-x^2} \frac{1}{2}x \, dy \, dx$$

$$m = \int_0^3 \frac{1}{2}xy\Big|_0^{9-x^2} dx \Rightarrow m = \int_0^3 \frac{1}{2}x(9-x^2) \, dx = \int_0^3 \left(\frac{9}{2}x - \frac{1}{2}x^3\right) dx$$

$$M = \left[\frac{9}{4}x^2 - \frac{1}{8}x^4\right]\Big|_0^3 = \frac{81}{4} - \frac{81}{8} = \underline{\underline{\frac{81}{8}}}$$

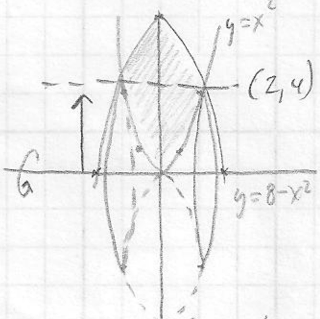
$$\bar{x} = \frac{8}{81}\int_0^3\int_0^{9-x^2}\int_0^1 x(xz) \, dz \, dy \, dx =$$

$$\bar{y} = \frac{8}{81}\int_0^3\int_0^{9-x^2}\int_0^1 yxz \, dz \, dy \, dx =$$

$$\bar{z} = \frac{8}{81}\int_0^3\int_0^{9-x^2}\int_0^1 (xz)z \, dz \, dy \, dx =$$

ps 15.6

40) $y = x^2$, $y = 8 - x^2$ @ x-axis



$\bar{y} = 4$ from symmetry

$A = \int_{-2}^{2} \int_{x^2}^{8-x^2} dy\, dx = \int_{-2}^{2} (8 - 2x^2)\, dx = 8x - \frac{2}{3}x^3 \Big|_{-2}^{2}$

$A = 2\left(16 - \frac{16}{3}\right) = \underline{\underline{\frac{64}{3}}}$

distance traveled by centroid = $2\pi(4)$ (circumferen of circle scribed by centroid)

$\therefore V = \frac{64}{3}(8\pi) = \underline{\underline{\frac{512\pi}{3}}}$

15.7

Triple Integrals in Cylindrical Coordinates

$$\int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\ r\ dz\ dr\ d\theta$$

To determine the limit, sketch the region G

① Identify the upper surface $z = g_2(r,\theta)$ & the lower surface $z = g_1(r,\theta)$

② make a 2-dimensional sketch of the projection R of the solid on the xy-plane. From this sketch r & θ limits of integration

From Rectangular to Cylindrical

$$\iiint_G f(x,y,z)\ dV = \underbrace{\iiint}_{\text{(appropriate limit)}} f(r\cos\theta, r\sin\theta, z)\ r\ dz\ dr\ d\theta$$

Triple Integral in spherical coordinates

$$\int_0^{\theta_0}\int_0^{\phi_0}\int_0^{\rho_0} f(\rho,\theta,\phi)\,\rho^2\sin\phi\ d\rho\ d\phi\ d\theta$$

From Rectangular to Sherical

$$\iiint_G f(x,y,z)\ dV = \underbrace{\iiint}_{\text{(appropriate limit)}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2\sin\phi\ d\rho\, d\phi\, d\theta$$

P. 1073

4) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a\sec\phi} \rho^2\sin\phi\ d\rho\ d\phi\ d\theta, \ (a>0)$

$\int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\rho^3\sin\phi\ \Big|_0^{a\sec\phi}\ d\phi\, d\theta$

$\frac{a^3}{3}\int_0^{2\pi}\int_0^{\pi/4} \frac{\sin\phi}{\cos^3\phi}\ d\phi\, d\theta \quad u = \cos\phi$

$du = -\sin\phi\ d\phi$

$-\frac{a^3}{3}\int_0^{2\pi} -\frac{1}{2}\sec^2\phi\ \Big|_0^{\pi/4}\ d\theta$

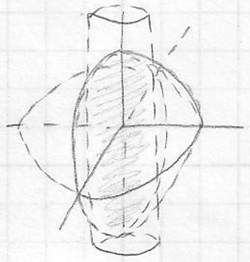
$\frac{a^3}{6}\int_0^{2\pi}(2-1)\ d\theta$

$\frac{a^3}{6}\int_0^{2\pi} d\theta$

$\frac{a^3}{6}\ \theta\Big|_0^{2\pi} = \underline{\underline{\frac{a^3\pi}{3}}}$

p.2 15.7

6)



$$V = 8 \int_0^{\pi/2} \int_0^{2} \int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$$

$\uparrow$ first octant

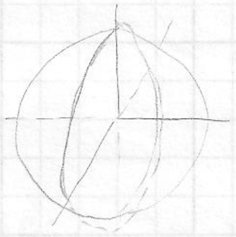
$$V = 8 \int_0^{\pi/2} \int_0^{2} r\sqrt{9-r^2}\,dr\,d\theta$$

$$V = -4 \int_0^{\pi/2} \frac{2}{3}(9-r^2)^{3/2}\Big|_0^2\,d\theta$$

$$V = \frac{8}{3}(27-5^{3/2}) \int_0^{\pi/2} d\theta$$

$$V = \frac{8}{3}(27-5\sqrt{5})\,\theta\Big|_0^{\pi/2} = \underline{\underline{\frac{4\pi}{3}(27-5\sqrt{5})}}$$

16) $\displaystyle\int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{k^2+y^2+z^2}\,dz\,dy\,d\theta$



$$\int_0^{2\pi} \int_0^{\pi} \int_0^{3} \rho(\varphi^2 \sin\phi)\,d\rho\,d\phi\,d\theta$$

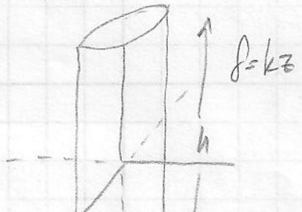
$$\int_0^{2\pi} \int_0^{\pi} \frac{1}{4}\rho^4 \sin\phi\Big|_0^3\,d\phi\,d\theta$$

$$\frac{81}{4} \int_0^{2\pi} \int_0^{\pi} \sin\phi\,d\phi\,d\theta$$

$$-\frac{81}{4} \int_0^{2\pi} \cos\phi\Big|_0^{\pi}\,d\theta$$

$$\frac{81}{2} \int_0^{2\pi} d\theta = \frac{81}{2}\theta\Big|_0^{2\pi} = \underline{\underline{81\pi}}$$

22)



$$M = \int_0^{2\pi} \int_0^{a} \int_0^{h} kz\,r\,dz\,dr\,d\theta$$

$$M = \frac{k}{2} \int_0^{2\pi} \int_0^{a} h^2\,r\,dr\,d\theta$$

$$M = \frac{kh^2}{2} \int_0^{2\pi} \frac{1}{2}r^2\Big|_0^a\,d\theta$$

$$M = \frac{kh^2a^2}{4} \int_0^{2\pi} d\theta$$

$$M = \frac{kh^2a^2}{4}\,\theta\Big|_0^{2\pi} \Rightarrow \underline{\underline{M = \frac{kh^2a^2\pi}{2}}}$$

p.3 15.7

26) centroid of solid

$$\bar{x} = \frac{1}{V}\int\int\int x\,dV \ , \ \bar{y} = \frac{1}{V}\iint\int y\,dV \ , \ \bar{z} = \frac{1}{V}\int\iint z\,dV$$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(2)^2(2) = \frac{8\pi}{3}$ $\qquad$ $\bar{x} = \bar{y} = 0$ by symmetry ; $z = r$

$$\bar{z} = \frac{3}{8\pi}\int_0^{2\pi}\int_0^{2}\int_r^{2} z\,r\,dz\,dr\,d\theta$$

$$\bar{z} = \frac{3}{8\pi}\int_0^{2\pi}\int_0^{2} \frac{1}{2}z^2 r \Big|_r^2 \ dr\,d\theta$$

$$\bar{z} = \frac{3}{16\pi}\int_0^{2\pi}\int_0^{2} (4r - r^3)\,dr\,d\theta$$

$$\bar{z} = \frac{3}{16\pi}\int_0^{2\pi} (2r^2 - \frac{1}{4}r^4)\Big|_0^2 \ d\theta$$

$$\bar{z} = \frac{3}{16\pi}\int_0^{2\pi}\left[\,4\,\right] d\theta \Rightarrow \bar{z} = \frac{3}{4\pi}\int_0^{2\pi} d\theta \Rightarrow \bar{z} = \frac{3}{4\pi}\,\theta\Big|_0^{2\pi} = \frac{3}{2} \ \therefore \ \underline{(0, 0, \tfrac{3}{2})}$$

28) above by $\rho = 4$, below by $\phi = \pi/3$; $\bar{x} = \bar{y} = 0$ by symmetry of solid

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3}\rho^3 \sin\phi\Big|_0^4 \ d\phi\,d\theta$$

$$V = \frac{64}{3}\int_0^{2\pi}\int_0^{\pi/3} \sin\phi \ d\phi\,d\theta = \frac{64}{3}\int_0^{2\pi} -\cos\phi\Big|_0^{\pi/3} d\theta = \frac{32}{3}\int_0^{2\pi} d\theta$$

$$V = \frac{64\pi}{3}$$

$$\bar{z} = \frac{3}{64\pi}\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4} z\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$\bar{z} = \frac{3}{64\pi}\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4} \rho^3 \cos\phi \sin\phi\,d\rho\,d\phi\,d\theta$$

$$\bar{z} = \frac{3}{64\pi}\int_0^{2\pi}\int_0^{\pi/3} \frac{1}{4}\rho^4\Big|_0^4 \cos\phi\sin\phi\,d\phi\,d\theta$$

$$\bar{z} = \frac{3}{\pi}\int_0^{2\pi}\int_0^{\pi/3} \cos\phi \sin\phi \ d\phi\,\rho\theta$$

$$\bar{z} = \frac{3}{\pi}\int_0^{2\pi} \frac{1}{2}\sin^2\phi\Big|_0^{\pi/3} \ d\theta$$

$$\bar{z} = \frac{9}{8\pi}\int_0^{2\pi} d\theta$$

$$\bar{z} = \frac{9}{4} \qquad \therefore \ \underline{(0, 0, \tfrac{9}{4})}$$

p.4 15.7

37)

$$m = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \delta_0 e^{-(\rho/R)^3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$u = -\frac{\rho^3}{R^3}$

$du = -\frac{3\rho^2}{R^3} d\rho$

$$m = -\frac{R^3\delta_0}{3}\int_0^{2\pi}\int_0^{\pi} e^{-(\rho/R)^3}\Big|_0^R \sin\phi \, d\phi \, d\theta$$

$$m = -\frac{R^3\delta_0}{3}\int_0^{2\pi}\int_0^{\pi} (e^{-1}-1) \sin\phi \, d\phi \, d\theta$$

$$m = \frac{R^3\delta_0(e^{-1}-1)}{3}\int_0^{2\pi} \cos\phi\Big|_0^{\pi} d\theta$$

$$m = \frac{2R^3\delta_0}{3}(1-e^{-1})\int_0^{2\pi} d\theta = \underline{\underline{\frac{4\pi R^3\delta_0}{3}(1-e^{-1})}}$$

40) I_y for $x^2+y^2 \le a^2$, $0 \le z \le h$; δ = constant

$$I_y = \int_0^{2\pi}\int_0^{a}\int_0^{h} (r^2\cos^2\theta + z^2)\delta \, r \, dz \, dr \, d\theta$$

$$I_y = \delta\int_0^{2\pi}\int_0^{a} zr^3\cos^2\theta + \frac{1}{3}z^3 r \Big|_0^h dr \, d\theta$$

$$I_y = \delta\int_0^{2\pi}\int_0^{a} \left(hr^3\cos^2\theta + \frac{1}{3}h^3 r\right) dr \, d\theta$$

$$I_y = \delta\int_0^{2\pi}\int_0^{a} \left(\frac{1}{4}hr^4\cos^2\theta + \frac{1}{6}h^3r^2\right)\Big|_0^a d\theta$$

$$I_y = \delta\int_0^{2\pi} \left(\frac{a^4}{4}h\cos^2\theta + \frac{a^2}{6}h^3\right) d\theta$$

$$I_y = \delta\int_0^{2\pi} \left[\frac{1}{8}a^4h(1+\cos 2\theta) + \frac{1}{6}a^2h^3\right] d\theta$$

$$I_y = \delta\left[\frac{1}{8}a^4h\left(\theta + \frac{1}{2}\sin 2\theta\right) + \frac{1}{6}a^2h^3\theta\right]\Big|_0^{2\pi}$$

$$I_y = \delta\left[\frac{1}{\cancel{8}_4}a^4h(\cancel{2}\pi + 0) + \frac{1}{\cancel{6}_3}a^2h^3(\cancel{2}\pi)\right]$$

$$I_y = \delta\left[\frac{1}{4}a^4h\pi + \frac{1}{3}a^2h^3\pi\right] = \underline{\underline{\delta a^2h\pi\left[\frac{3a^2+4h^2}{12}\right]}}$$

Ch.15

15.8 Jacobian

Consider parametric eq'ns of the form $x = x(u,v)$, $y = y(u,v)$

Parametrics of this type associate points in the xy-plane to points in the uv-plane. They can be written as

$$\vec{r} = \vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j}$$

T is called a transformation from the uv-plane to the xy-plane & (x,y) the image of (u,v) under the transformation T. Also, T maps (u,v) into (x,y). The set R of all images in the xy-plane of a set S in the uv-plane is called the image of S under T. If distinct points in the uv-plane have distinct images in the xy-plane, then T is one-to-one

If we define a transformation from the xy-plane to the uv-plane that maps the image of (u,v) under T back into (u,v), denoted by T^{-1}, it is called the inverse of T.

If T is the transformation from the uv-plane to the xy-plane def'd by the eq'ns $x = x(u,v)$ and $y = y(u,v)$, then the Jacobian of T is denoted by $J(u,v)$ or by $\frac{\partial(x,y)}{\partial(u,v)}$ and is defined by

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \cdot \frac{\partial x}{\partial v}$$

$\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$; this tells us that for small values of Δu & Δv, the area R is approximately the absolute value of the Jacobian ~~time~~ the area of S.

Change of variables for Double Integrals

$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}$$

If T is the transformation from uvw-space to xyz-space def'd by the eq'ns $x = x(u,v,w)$, $y = y(u,v,w)$, & $z = z(u,v,w)$, then the Jacobian of T is denoted by $J(u,v,w)$ or $\partial(x,y,z)/\partial(u,v,w)$ and is def'd by

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad ; \Delta V \approx \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \Delta u \Delta v \Delta w$$

15.8

Change of Variable for Triple Integral

$$\iiint_R f(x,y,z)\, dV_{xyz} = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV_{uvw}$$

From 15.7, eqn (6) & (10) becomes, respectively,

$$\iiint_G f(x,y,z)\, dV = \underset{\substack{\text{appropriate} \\ \text{limit}}}{\iiint} f(r\cos\theta, r\sin\theta, z) \frac{\partial(x,y,z)}{\partial(u,v,z)}\, dz\, dr\, d\theta$$

&

$$\iiint_G f(x,y,z)\, dV = \underset{\substack{\text{appropriate} \\ \text{limit}}}{\iiint} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \frac{\partial(x,y,z)}{\partial(u,v,w)}\, d\rho\, d\phi\, d\theta$$

P.1085

7) $u = x^2 - y^2$, $v = x^2 + y^2$, $(x > 0, y > 0)$

$\frac{u+v}{2} = x^2 \quad \frac{v-u}{2} = y^2$

$\frac{\sqrt{u+v}}{\sqrt{2}} = x \quad \frac{\sqrt{v-u}}{\sqrt{2}} = y$

$$J(u,v) = \begin{vmatrix} \frac{1}{2\sqrt{2}\sqrt{u+v}} & \frac{1}{2\sqrt{2}\sqrt{u+v}} \\ \frac{-1}{2\sqrt{2}\sqrt{v-v}} & \frac{1}{2\sqrt{2}\sqrt{v-u}} \end{vmatrix} = \frac{1}{8\sqrt{v^2-u^2}} + \frac{1}{8\sqrt{v^2-u^2}} = \frac{2}{8\sqrt{v^2-u^2}} = \underline{\underline{\frac{1}{4\sqrt{v^2-u^2}}}}$$

22) $\iint_R e^{-(x^2+4y^2)}\, dA$; $\frac{x^2}{4} + y^2 = 1 \quad \frac{x}{2} = u \Rightarrow x = 2u \quad \frac{y}{1} = v \Rightarrow y = v$

$$\iint_S e^{-(4u^2+4v^2)} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA \qquad J(u,v) = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2$$

$$\iint_S 2e^{-(4u^2+4v^2)}\, du\, dv$$

$$2\int_0^{2\pi}\int_0^1 e^{-4r^2}\, r\, dr\, d\theta$$

$$-\frac{1}{4}\int_0^{2\pi} e^{-4r^2}\Big|_0^1\, d\theta$$

$$-\frac{1}{4}(e^{-4}-1)\int_0^{2\pi} d\theta$$

$$-\frac{1}{4}(e^{-4}-1)\,2\pi$$

$$\underline{\underline{\frac{1}{2}\pi(1-e^{-4})}}$$