Ch. 4 Integration

sec. 4.2 Area

Sigma Notation $\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$

$i$ is the index of summation

$a_i$ is the $i^{th}$ term

1 is the lower bound

$n$ is the upper bound

Summation Formulas

1) $\sum_{i=1}^{n} c = cn$ 2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ 3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ 4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Area $= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$ $\Delta x = \frac{b-a}{n}$ , $x_i = a + i\Delta x$

P.267

18) $\sum_{i=1}^{10}(i-1) = \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1$

$= \frac{10(10+1)}{2} - 10$

$= 55 - 10 = \underline{\underline{45}}$

22) $s(n) = \left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right)$

$\lim_{n\to\infty}\left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right) = \underline{\underline{\frac{8}{3}}}$

32) $\lim_{n\to\infty} \sum_{i=1}^{n}\left(1 + \frac{i}{n}\right)\left(\frac{2}{n}\right)$

$\lim_{n\to\infty} \sum_{1}^{n}\left(1 + \frac{1}{n}\left(\frac{n(n+1)}{2}\right)\right)\left(\frac{2}{n}\right)$

$\lim_{n\to\infty} \sum_{1}^{n}\left(\frac{2}{n} + 1 + \frac{1}{n}\right) = \underline{\underline{1}}$

42) $y = 3x - 4$ $[2,5]$ $\Delta x = \frac{5-2}{n} = \frac{3}{n}$

$x_i = 2 + \frac{3i}{n}$

$A = \lim_{n\to\infty} \sum_{1}^{n} 3\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$

$A = \lim_{n\to\infty} \sum_{1}^{n} \frac{18}{n} + \frac{27}{n^2}\left(\frac{n(n+1)}{2}\right)$

$A = \lim_{n\to\infty} \sum_{1}^{n} \frac{18}{n} + \frac{27}{2} + \frac{27}{2n} = \underline{\underline{\frac{27}{2}}}$

46) $y = 2x - x^3$ $[0,1]$ $\Delta x = \frac{1-0}{n} = \frac{1}{n}$

$x_i = 0 + i\Delta x = \frac{i}{n}$

$A = \lim_{n\to\infty} \sum_{1}^{n}\left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left[\frac{1}{n}\right]$

$A = \lim_{n\to\infty} \sum_{1}^{n}\left[\frac{2}{n^2} i - \frac{1}{n^4} i^3\right]$

$A = \lim_{n\to\infty} \sum_{1}^{n}\left[\frac{8}{n^2}\left(\overset{n^2+n}{\frac{n(n+1)}{8}}\right) - \frac{1}{n^4}\left(\overset{n^4+2n^3+n^2}{\frac{n^2(n+1)^2}{4}}\right)\right]$

$A = \lim_{n\to\infty} \sum_{1}^{n} 1 + \frac{1}{n} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2} = \underline{\underline{\frac{3}{4}}}$

sec. 4.3 Riemann Sums, Definite Integrals

Riemann Sum $\sum_1^n f(c_i)\Delta x_i$ ; $x_{i-1} \le c_i \le x_i$

Regular partition $\|\Delta\| = \Delta x = \frac{b-a}{n}$

Integral $\lim_{\Delta x \to 0} \sum_1^n f(c_i)\Delta x_i = \int_a^b f(x)\,dx$

This limit is called the definite integral of f from a to b.

Continuity implies Integrability

If a fn f is continuous on the closed interval $[a,b]$, then f is integrable on $[a,b]$

Area $= \int_a^b f(x)\,dx$

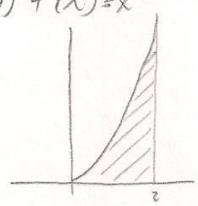
Properties of Integrals

① $\int_a^a f(x)\,dx = 0$ ② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ ③ $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, $a < c < b$

④ $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$ ⑤ $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

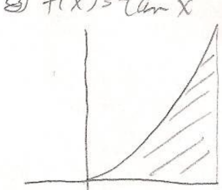
P.277

4) $f(x) = x^2$



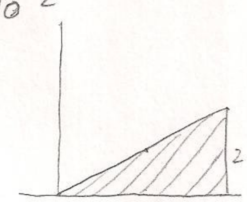

$A = \int_0^2 x^2\,dx$

8) $f(x) = \tan x$



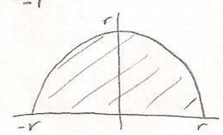

$A = \int_0^{\pi/4} \tan x\,dx$

14) $\int_0^4 \frac{1}{2}x\,dx$




$A = \frac{1}{2}bh = \frac{1}{2}(4)(2) = \underline{\underline{4}}$

20) $\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$




$A = \frac{1}{2}\pi r^2$

28) $\int_0^1 x^3\,dx$ $\quad \Delta x = \frac{1}{n}$

$x_i = \frac{i}{n}$

$\lim_{n\to\infty} \sum_1^n \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right)$

$\lim_{n\to\infty} \sum_1^n \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = \underline{\underline{\frac{1}{4}}}$ ($n^2(n+1)^2 = n^4 + 2n^3 + n^2$)

## Sec 4.4 The Fundamental Theorem of Calculus

If a function $f$ is continuous on the closed interval $[a, b]$, and $F$ is the antiderivative of $f$ on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Mean Value Theorem for Integrals
If $f$ is continuous on the closed interval $[a, b]$, then there exists a number $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Average Value of a function on an interval

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

The Second Fundamental Theorem of Calculus
If $f$ is continuous on an open interval $I$ containing $a$, then for every $x$ in the interval

$$\frac{d}{dx}\left[\int_a^b f(t)\,dt\right] = f(x)$$

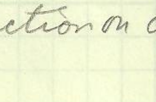
P 289

8) $\int_{-1}^{1} (t^3 - 9t)\,dt$

$\frac{1}{4}t^4 - \frac{9}{2}t^2 \Big|_{-1}^{1}$

$\left(\frac{1}{4} - \frac{9}{2}\right) - \left(\frac{1}{4} - \frac{9}{2}\right) = \underline{\underline{0}}$

20) $\int_0^4 |x^2 - 4x + 3|\,dx$ $(x-1)(x-3)$




$\int_0^1 x^2 - 4x + 3\,dx$

$+ \int_1^3 -x^2 + 4x - 3\,dx + \int_3^4 x^2 - 4x + 3\,dx$

$\frac{1}{3}x^3 - 2x^2 + 3x\Big|_0^1 - \frac{1}{3}x^3 + 2x^2 - 3x\Big|_1^3 + \frac{1}{3}x^3 - 2x^2 + 3x\Big|_3^4$

$\left(\frac{1}{3} - 2 + 3\right) - (0) + (\cancel{-9} + 12 - 3)^0 - \left(-\frac{1}{3} + 2 - 3\right) + \left(\left(\frac{64}{3} - 32 + 12\right) - \left(\cancel{9 - 18 + 3}\right)^0\right)$

$\frac{1}{3} + 1 + \frac{1}{3} + 1 + \frac{64}{3} - 20$

$24 - 20$

$\underline{\underline{4}}$

24) $\int_{\pi/4}^{\pi/2} (2 - \csc^2 x)\,dx$

$2x + \cot x \Big|_{\pi/4}^{\pi/2}$

$(\pi + 1) - \left(\frac{\pi}{2} + 1\right) = \underline{\underline{\frac{\pi}{2}}}$

28) $P = \frac{2}{\pi}\int_0^{\pi/2} \sin\theta\,d\theta$

$P = \frac{2}{\pi}(-\cos\theta)\Big|_0^{\pi/2}$

$P = \frac{2}{\pi}(0 + 1) = \underline{\underline{\frac{2}{\pi}}}$

32) $y = \frac{1}{x^2} = x^{-2}$




$A = \int_1^2 x^{-2}\,dx$

$A = -\frac{1}{x}\Big|_1^2$

$A = -\frac{1}{2} + 1 = \underline{\underline{\frac{1}{2}}}$

P.290

62) $F(x) = \int_0^x \tan^4 x \, dx$

$F'(x) = \frac{d}{dx}\left[\int_0^x \tan^4 t \, dt\right]$

$F'(x) = \tan^4 x$

64) $F(x) = \int_1^x \frac{t^2}{t^2+1} \, dt$

$F'(x) = \frac{x^2}{x^2+1}$

66) $\int_{-x}^x t^3 \, dt$

$\int_{-x}^0 t^3 \, dt + \int_0^x t^3 \, dt$

$-\int_0^{-x} t^3 \, dt + \int_0^x t^3 \, dt$

$-\frac{1}{4} t^4 \Big|_0^{-x} + \frac{1}{4} t^4 \Big|_0^x$

$-\frac{1}{4} x^4 + \frac{1}{4} x^4 = \underline{\underline{0}}$

68) $F(x) = \int_2^{x^2} \frac{1}{t^2} \, dt$

$u = x^2$

$du = 2x \, dx$

$F'(x) = \frac{dF}{du} \cdot \frac{du}{dx}$ (chain rule)

$F'(x) = \frac{1}{u^2} \cdot 2x$

$F'(x) = \frac{1}{(x^2)^2} \cdot 2x = \underline{\underline{\frac{2}{x^3}}}$

73) $C(x) = 5000(25 + 3\int_0^x t^{1/4} \, dt)$

$C(x) = 125000 + 15000 \int_0^x t^{1/4} \, dt$

$C(x) = 125,000 + 15000 \cdot \frac{4}{5} t^{5/4} \Big|_0^x$

$\underline{\underline{C(x) = 125,000 + 12,000 x^{5/4}}}$

$C(1) = 125,000 + 12,000 = \underline{\underline{\$137,000}}$

$C(5) = 125,000 + 12,000(5)^{5/4} \approx \underline{\underline{\$214,720.93}}$

$C(10) = 125,000 + 12,000(10)^{5/4} \approx \underline{\underline{\$338,393.53}}$

## Sec. 4.5 Integration by Substitution

**Integration of Composite Functions**

Let $g$ be a function whose range is an interval $I$, and let $f$ be a function that is continuous on $I$. If $g$ is differentiable on its domain and $F$ is the antiderivative of $f$ on $I$, then

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$$

If $u = g(x)$, then $du = g'(x)\,dx$ and $\int f(u)\,du = F(u) + C$

**Integration of Even and Odd Functions**

Let $f$ be integrable on the closed interval $[-a, a]$

1) If $f$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

2) If $f$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

P.301

8) $\int \underbrace{(x^2-1)}_{u}\underbrace{(2x)\,dx}_{du}$ $\quad u = x^2-1$, $du = 2x\,dx$

$\int u\,du$

$\frac{1}{2}u^2 + C$

$\frac{1}{2}(x^2-1)^2 + C$

12) $\int x(4x^2+3)\,dx$ $\quad u = 4x^2+3$, $du = 8x\,dx$

$\frac{1}{8}\int \underbrace{(4x^2+3)}_{u}\underbrace{(8x\,dx)}_{du}$

$\frac{1}{8}\int u\,du$

$\frac{1}{8}\cdot\frac{1}{2}u^2 + C$

$\frac{1}{16}(4x^2+3)^2 + C$

32) $\int x\sin x^2\,dx$ $\quad u = x^2$, $du = 2x\,dx$

$\frac{1}{2}\int \sin \underbrace{x^2}_{u}\,\underbrace{2x\,dx}_{du}$

$\frac{1}{2}\int \sin u\,du$

$-\frac{1}{2}\cos u + C \Rightarrow -\frac{1}{2}\cos x^2 + C$

36) $-1\int \sec \underbrace{(1-x)}_{u}\tan\underbrace{(1-x)}_{u}\underbrace{(-dx)}_{du}$ $\quad u = 1-x$, $du = -dx$

$\int \sec u \tan u\,du$

$\sec u + C$

$\sec(1-x) + C$

48) $\int \frac{2x-1}{\sqrt{x+3}}\,dx$ $\quad u = x+3$, $du = dx$ $\quad u-3 = x$, $2(u-3) = 2x$, $2(u-3)-1 = 2x-1$, $2u-7 = 2x-1$

$\int \frac{2u-7}{u^{1/2}}\,du$

$\int \frac{2u}{u^{1/2}} - \frac{7}{u^{1/2}}\,dx$

$\int 2u^{1/2} - 7u^{-1/2}\,dx$

$2\cdot\frac{2}{3}u^{3/2} - 7\cdot\frac{2}{1}u^{1/2} + C$

$\frac{4}{3}(x+3)^{3/2} - 14(x+3)^{1/2} + C$

64) $\frac{1}{2}\int_{\pi/12}^{\pi/4} \csc \underbrace{2x}_{u} \cot \underbrace{2x}_{u}\,\underbrace{2dx}_{du}$ $\quad u = 2x$, $du = 2dx$

$\frac{1}{2}\int \csc u \cot u\,du$

$-\frac{1}{2}\csc u + C$

$-\frac{1}{2}\csc 2x \Big|_{\pi/12}^{\pi/4} = -\frac{1}{2}\left[\left(\csc\frac{\pi}{2} - \csc\frac{\pi}{6}\right)\right]$

$= -\frac{1}{2}(1-2) = \frac{1}{2}$